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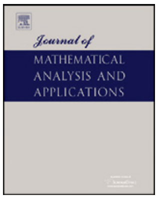

Correctors and field fluctuations for the $p_\epsilon(x)$-Laplacian with rough exponents ☆

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ABSTRACT

We provide a corrector theory for the strong approximation of fields inside composites made from two materials with different power-law behavior. The correctors are used to develop bounds on the local singularity strength for gradient fields inside micro-structured media. The bounds are multi-scale in nature and can be used to measure the amplification of applied macroscopic fields by the micro-structure.

1. Introduction

In this article we consider boundary value problems associated with fields inside heterogeneous materials made from two power-law materials. The geometry of the composite is periodic and is specified by the indicator function of the sets occupied by each of the materials. The indicator functions of material one and two are denoted by χ_1 and χ_2, where $\chi_1(y) = 1$ in material one and is zero outside and $\chi_2(y) = 1 - \chi_1(y)$. The constitutive law for the heterogeneous medium is described by $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$,

$$A(y, \xi) = \sigma(y)|\xi|^{p(y)-2}\xi, \tag{1.1}$$

with $\sigma(y) = \chi_1(y)\sigma_1 + \chi_2(y)\sigma_2$, and $p(y) = \chi_1(y)p_1 + \chi_2(y)p_2$, periodic in y, with unit period cell $Y = (0, 1)^n$. This simple constitutive model is used in the mathematical description of many physical phenomena including plasticity [17,18,20,10], nonlinear dielectrics [9,8,12,21,22], and fluid flow [19,2]. We study the problem of periodic homogenization associated with the solutions u_ϵ to the problems

$$-\mathrm{div}\left(A\left(\frac{x}{\epsilon}, \nabla u_\epsilon\right)\right) = f \quad \text{on } \Omega,\ u_\epsilon \in W_0^{1,p_1}(\Omega), \tag{1.2}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $2 \leqslant p_1 \leqslant p_2$, $f \in W^{-1,q_2}(\Omega)$, and $1/p_1 + 1/q_2 = 1$. The differential operator appearing on the left-hand side of (1.2) is commonly referred to as the $p_\epsilon(x)$-Laplacian. For the case at hand, the exponents $p(x)$ and coefficients $\sigma(x)$ are taken to be simple functions. Because the level sets associated with these functions can

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be quite general and irregular they are referred to as rough exponents and coefficients. In this context all solutions are understood in the usual weak sense [26].

One of the basic problems in homogenization theory is to understand the asymptotic behavior as $\epsilon \to 0$, of the solutions u_ϵ to the problems (1.2). It was proved in [26] that $\{u_\epsilon\}_{\epsilon>0}$ converges weakly in $W^{1,p_1}(\Omega)$ to the solution u of the *homogenized* problem

$$-\mathrm{div}\big(b(\nabla u)\big) = f \quad \text{on } \Omega,\ u \in W_0^{1,p_1}(\Omega), \tag{1.3}$$

where the monotone map $b : \mathbb{R}^n \to \mathbb{R}^n$ (independent of f and Ω) can be obtained by solving an auxiliary problem for the operator (1.2) on a periodicity cell.

The notion of homogenization is intimately tied to the Γ-convergence of a suitable family of energy functionals I_ϵ as $\epsilon \to 0$ [5,26]. Here the connection is natural in that the family of boundary value problems (1.3) corresponds to the Euler equations of the associated energy functionals I_ϵ and the solutions u_ϵ are their minimizers. The homogenized solution is precisely the minimizer of the Γ-limit of the sequence $\{I_\epsilon\}_{\epsilon>0}$. The connections between Γ limits and homogenization for the power-law materials studied here can be found in [26]. The explicit formula for the Γ-limit of the associated energy functionals for layered materials was obtained recently in [16].

Homogenization theory relates the average behavior seen at large length scales to the underlying heterogeneous structure. It allows one to approximate $\{\nabla u_\epsilon\}_{\epsilon>0}$ in terms of ∇u, where u is the solution of the homogenized problem (1.3). The homogenization result given in [26] shows that the average of the error incurred in this approximation of ∇u_ϵ decays to 0.

On the other hand it is well known [11] that the presence of large local fields either electric or mechanical often precedes the onset of material failure. For composite materials the presence of the heterogeneity can amplify the applied load and generate local fields with very high intensities. The goal of the analysis presented here is to develop tools for quantifying the effect of load transfer between length scales inside heterogeneous media. In this article we provide methods for quantitatively measuring the excursions of local fields generated by applied loads. We present a new corrector result that delivers an approximation to ∇u_ϵ up to an error that converges to zero strongly in the norm. Our approach delivers strong approximations for the gradients inside each phase, see Section 2.2.1.

The strong approximations are used to develop new tools that provide lower bounds on the local gradient field intensity inside micro-structured media. The bounds are expressed in terms of the L^q norms of gradients of the solutions of the local corrector problems. These results provide a lower bound on the amplification of the macroscopic (average) gradient field by the micro-structure. The bounds are shown to hold for every q for which the gradient of the corrector is L^q integrable, see Section 2.2.2. The critical values of q for which these moments diverge provide lower bounds on the L^q integrability of the gradients ∇u_ϵ when ϵ is sufficiently small. In [13], similar lower bounds are established for field concentrations for mixtures of linear electrical conductors in the context of two scale convergence.

The corrector results are presented for layered materials and for dispersions of inclusions embedded inside a host medium. For the dispersed micro-structures the included material is taken to have the lower power-law exponent than that of the host phase. For both of these cases it is shown that the homogenized solution lies in $W_0^{1,p_2}(\Omega)$. We use this higher order integrability to provide an algorithm for building correctors and construct a sequence of strong approximations to the gradients inside each material, see Theorem 2.6. When the host phase has a lower power-law exponent than the included phase one can only conclude that the homogenized solution lies in $W_0^{1,p_1}(\Omega)$ and the techniques developed here do not apply.

The earlier work of [6] provides the corrector theory for homogenization of monotone operators that in our case applies to composite materials made from constituents having the same power-law growth but with rough coefficients $\sigma(x)$. The corrector theory for monotone operators with uniform power-law growth is developed further in [7], where it is used to extend multi-scale finite element methods to nonlinear equations for stationary random media. Recent work considers the homogenization of $p_\epsilon(x)$-Laplacian boundary value problems for smooth exponential functions $p_\epsilon(x)$ uniformly converging to a limit function $p_0(x)$ [1]. There the convergence of the family of solutions for these homogenization problems is expressed in the topology of $L^{p_0(\cdot)}(\Omega)$ [1].

The paper is organized as follows. In Section 2, we state the problem and formulate the main results. Section 3 contains the proof of the properties of the homogenized operator. Section 4 is devoted to proving the higher order integrability of the homogenized solution. Section 5 contains lemmas and integral inequalities for the correctors used to prove the main results. Section 6 contains the proof of the main results.

2. Statement of the problem and main results

2.1. Notation

In this paper we consider two nonlinear power-law materials periodically distributed inside a domain $\Omega \subset \mathbb{R}^n$. The periodic mixture is described as follows. We introduce the unit period cell $Y = (0,1)^n$ of the micro-structure. Let F be an open subset of Y of material one, with smooth boundary ∂F, such that $\overline{F} \subset Y$. The function $\chi_1(y) = 1$ inside F and 0 outside and $\chi_2(y) = 1 - \chi_1(y)$. We extend $\chi_1(y)$ and $\chi_2(y)$ by periodicity to $\mathbb{R}^n$ and the ϵ-periodic mixture inside Ω is described by the oscillatory characteristic functions $\chi_1^\epsilon(x) = \chi_1(x/\epsilon)$ and $\chi_2^\epsilon(x) = \chi_2(x/\epsilon)$. Here we will consider the

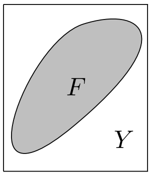

Fig. 1. Unit cell: Dispersed micro-structure.

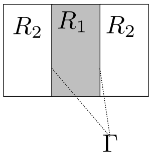

Fig. 2. Unit cell: Layered material.

case where F is given by a simply connected inclusion embedded inside a host material (see Fig. 1). A distribution of such inclusions is commonly referred to as a periodic dispersion of inclusions.

In this article we also consider layered materials. For this case the representative unit cell consists of a layer of material one, denoted by R_1, sandwiched between layers of material two, denoted by R_2. The interior boundary of R_1 is denoted by Γ. Here $\chi_1(y) = 1$ for $y \in R_1$ and 0 in R_2, and $\chi_2(y) = 1 - \chi_1(y)$ (see Fig. 2).

On the unit cell Y, the constitutive law for the nonlinear material is given by (1.1) with exponents p_1 and p_2 satisfying $2 \leqslant p_1 \leqslant p_2$. Their Hölder conjugates are denoted by $q_2 = p_1/(p_1 - 1)$ and $q_1 = p_2/(p_2 - 1)$ respectively. For $i = 1, 2$, $W^{1,p_i}_{per}(Y)$ denotes the set of all functions $u \in W^{1,p_i}(Y)$ with mean value zero that have the same trace on the opposite faces of Y. Each function $u \in W^{1,p_i}_{per}(Y)$ can be extended by periodicity to a function of $W^{1,p_i}_{loc}(\mathbb{R}^n)$.

The Euclidean norm and the scalar product in $\mathbb{R}^n$ are denoted by $|\cdot|$ and $(\cdot,\cdot)$, respectively. If $A \subset \mathbb{R}^n$, $|A|$ denotes the Lebesgue measure and $\chi_A(x)$ denotes its characteristic function.

The constitutive law for the ϵ-periodic composite is described by $A_\epsilon(x, \xi) = A(x/\epsilon, \xi)$, for every $\epsilon > 0$, for every $x \in \Omega$, and for every $\xi \in \mathbb{R}^n$.

A calculation shows [3] that there exist constants $C_1, C_2 > 0$ such that for almost every $x \in \mathbb{R}^n$ and for every $\xi \in \mathbb{R}^n$, A satisfies the following:

(1) For all $\xi \in \mathbb{R}^n$, $A(\cdot, \xi)$ is Y-periodic and Lebesgue measurable.
(2) $|A(y, 0)| = 0$ for all $y \in \mathbb{R}^n$.
(3) Continuity

$$\bigl|A(y,\xi_1) - A(y,\xi_2)\bigr| \leqslant C_1\bigl[\chi_1(y)|\xi_1 - \xi_2|\bigl(1 + |\xi_1| + |\xi_2|\bigr)^{p_1-2} + \chi_2(y)|\xi_1 - \xi_2|\bigl(1 + |\xi_1| + |\xi_2|\bigr)^{p_2-2}\bigr]. \tag{2.1}$$

(4) Monotonicity

$$\bigl(A(y,\xi_1) - A(y,\xi_2), \xi_1 - \xi_2\bigr) \geqslant C_2\bigl(\chi_1(y)|\xi_1 - \xi_2|^{p_1} + \chi_2(y)|\xi_1 - \xi_2|^{p_2}\bigr). \tag{2.2}$$

2.2. Dirichlet boundary value problem

We shall consider the following Dirichlet boundary value problem

$$\begin{cases} -\operatorname{div}\bigl(A_\epsilon(x, \nabla u_\epsilon)\bigr) = f \quad \text{on } \Omega, \\ u_\epsilon \in W^{1,p_1}_0(\Omega), \end{cases} \tag{2.3}$$

where $f \in W^{-1,q_2}(\Omega)$.

The following homogenization result holds.

Theorem 2.1 *(Homogenization theorem). (See [26].) As $\epsilon \to 0$, the solutions u_ϵ of* (2.3) *converge weakly to u in $W^{1,p_1}(\Omega)$, where u is the solution of*

$$-\operatorname{div}\bigl(b(\nabla u)\bigr) = f \quad \textit{on } \Omega, \tag{2.4}$$

$$u \in W^{1,p_1}_0(\Omega); \tag{2.5}$$

and the function $b:\mathbb{R}^n\to\mathbb{R}^n$ *is defined for all* $\xi\in\mathbb{R}^n$ *by*

$$b(\xi)=\int_Y A\big(y,p(y,\xi)\big)\,dy, \tag{2.6}$$

where $p:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ *is defined by*

$$p(y,\xi)=\xi+\nabla\upsilon_\xi(y), \tag{2.7}$$

where υ_ξ *is the solution to the cell problem*:

$$\begin{cases}\displaystyle\int_Y\big(A(y,\xi+\nabla\upsilon_\xi),\nabla w\big)\,dy=0, & \text{for every } w\in W^{1,p_1}_{per}(Y),\\ \upsilon_\xi\in W^{1,p_1}_{per}(Y).\end{cases} \tag{2.8}$$

Remark 2.2. The following a priori bound is satisfied

$$\sup_{\epsilon>0}\bigg(\int_\Omega\chi_1^\epsilon(x)\big|\nabla u_\epsilon(x)\big|^{p_1}\,dx+\int_\Omega\chi_2^\epsilon(x)\big|\nabla u_\epsilon(x)\big|^{p_2}\,dx\bigg)\leqslant C<\infty, \tag{2.9}$$

where C does not depend on ϵ. The proof of this bound is given in Lemma 5.5.

Remark 2.3. The function b, defined in (2.6), satisfies the following properties for every $\xi_1,\xi_2\in\mathbb{R}^n$.

(1) Continuity: There exists a positive constant $\overline{C_1}$ such that

$$\begin{aligned}\big|b(\xi_1)-b(\xi_2)\big|\leqslant\overline{C_1}\big[&|\xi_1-\xi_2|^{\frac{1}{p_1-1}}\big(1+|\xi_1|^{p_1}+|\xi_2|^{p_1}+|\xi_1|^{p_2}+|\xi_2|^{p_2}\big)^{\frac{p_1-2}{p_1-1}}\\ &+|\xi_1-\xi_2|^{\frac{1}{p_2-1}}\big(1+|\xi_1|^{p_1}+|\xi_2|^{p_1}+|\xi_1|^{p_2}+|\xi_2|^{p_2}\big)^{\frac{p_2-2}{p_2-1}}\big].\end{aligned} \tag{2.10}$$

(2) Monotonicity: There exists a positive constant $\overline{C_2}$ such that

$$\begin{aligned}\big(b(\xi_1)-b(\xi_2),\xi_1-\xi_2\big)&\geqslant\overline{C_2}\bigg(\int_Y\chi_1(y)\big|p(y,\xi_1)-p(y,\xi_2)\big|^{p_1}\,dy+\int_Y\chi_2(y)\big|p(y,\xi_1)-p(y,\xi_2)\big|^{p_2}\,dy\bigg)\\ &\geqslant 0.\end{aligned} \tag{2.11}$$

Properties (2.10) and (2.11) are proved in Section 3.

Remark 2.4. Since the solution υ_ξ of (2.8) can be extended by periodicity to a function of $W^{1,p_1}_{loc}(\mathbb{R}^n)$, then (2.8) is equivalent to $-\operatorname{div}(A(y,\xi+\nabla\upsilon_\xi(y)))=0$ over $D'(\mathbb{R}^n)$, i.e.,

$$-\operatorname{div}\big(A\big(y,p(y,\xi)\big)\big)=0\quad\text{in } D'\big(\mathbb{R}^n\big)\text{ for every }\xi\in\mathbb{R}^n. \tag{2.12}$$

Moreover, by (2.8), we have

$$\int_Y\big(A\big(y,p(y,\xi)\big),p(y,\xi)\big)\,dy=\int_Y\big(A\big(y,p(y,\xi)\big),\xi\big)\,dy=\big(b(\xi),\xi\big). \tag{2.13}$$

For $\epsilon>0$, define $p_\epsilon:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ by

$$p_\epsilon(x,\xi)=p\bigg(\frac{x}{\epsilon},\xi\bigg)=\xi+\nabla\upsilon_\xi\bigg(\frac{x}{\epsilon}\bigg), \tag{2.14}$$

where υ_ξ is the unique solution of (2.8). The functions p and p_ϵ are easily seen to have the following properties

$$p(\cdot,\xi)\text{ is }Y\text{-periodic}\quad\text{and}\quad p_\epsilon(x,\xi)\text{ is }\epsilon\text{-periodic in }x, \tag{2.15}$$

$$\int_Y p(y,\xi)\,dy=\xi, \tag{2.16}$$

$$p_\epsilon(\cdot,\xi)\rightharpoonup\xi\quad\text{in } L^{p_1}\big(\Omega;\mathbb{R}^n\big)\text{ as }\epsilon\to 0, \tag{2.17}$$

$$p(y,0)=0 \quad \text{for almost every } y, \tag{2.18}$$

$$A\left(\frac{\cdot}{\epsilon}, p_\epsilon(\cdot,\xi)\right) \rightharpoonup b(\xi) \quad \text{in } L^{q_2}\big(\Omega;\mathbb{R}^n\big) \text{ as } \epsilon \to 0. \tag{2.19}$$

We now state the higher order integrability properties of the homogenized solution for periodic dispersions of inclusions and layered micro-geometries.

Theorem 2.5. *Given a periodic dispersion of inclusions or a layered material then the solution u of* (2.4) *belongs to $W_0^{1,p_2}(\Omega)$.*

The proof of this theorem is given in Section 4.

2.2.1. Statement of the corrector theorem

We now describe the family of correctors that provide a strong approximation of the sequence $\{\chi_i^\epsilon \nabla u_\epsilon\}_{\epsilon>0}$ in the $L^{p_i}(\Omega,\mathbb{R}^n)$ norm. We denote the rescaled period cell with side length $\epsilon>0$ by Y_ϵ and write $Y_\epsilon^i = \epsilon i + Y_\epsilon$, where $i \in \mathbb{Z}^n$. In what follows it is convenient to define the index set $I_\epsilon = \{i \in \mathbb{Z}^n \colon\ Y_\epsilon^i \subset \Omega\}$. For $\varphi \in L^{p_2}(\Omega;\mathbb{R}^n)$, we define the local average operator M_ϵ associated with the partition Y_ϵ^i, $i \in I_\epsilon$ by

$$M_\epsilon(\varphi)(x) = \begin{cases} \sum_{i\in I_\epsilon} \chi_{Y_\epsilon^i}(x) \frac{1}{|Y_\epsilon^i|} \int_{Y_\epsilon^i} \varphi(y)\,dy, & \text{if } x \in \bigcup_{i\in I_\epsilon} Y_\epsilon^i, \\ 0, & \text{if } x \in \Omega \setminus \bigcup_{i\in I_\epsilon} Y_\epsilon^i. \end{cases} \tag{2.20}$$

The family M_ϵ has the following properties:

(1) For $i=1,2$, $\|M_\epsilon(\varphi)-\varphi\|_{L^{p_i}(\Omega;\mathbb{R}^n)} \to 0$ as $\epsilon \to 0$ (see [23]).
(2) $M_\epsilon(\varphi) \to \varphi$ a.e. on Ω (see [23]).
(3) From Jensen's inequality we have $\|M_\epsilon(\varphi)\|_{L^{p_i}(\Omega;\mathbb{R}^n)} \leqslant \|\varphi\|_{L^{p_i}(\Omega;\mathbb{R}^n)}$, for every $\varphi \in L^{p_2}(\Omega;\mathbb{R}^n)$ and $i=1,2$.

The strong approximation to the sequence $\{\chi_i^\epsilon \nabla u_\epsilon\}_{\epsilon>0}$ is given by the following corrector theorem.

Theorem 2.6 (*Corrector theorem*). *Let $f \in W^{-1,q_2}(\Omega)$, let u_ϵ be the solutions to the problem* (2.3), *and let u be the solution to problem* (2.4). *Then, for periodic dispersions of inclusions and for layered materials, we have*

$$\int\limits_\Omega \big|\chi_i^\epsilon(x) p_\epsilon\big(x, M_\epsilon(\nabla u)(x)\big) - \chi_i^\epsilon(x)\nabla u_\epsilon(x)\big|^{p_i}\,dx \to 0, \tag{2.21}$$

as $\epsilon \to 0$, for $i=1,2$.

The proof of Theorem 2.6 is given in Section 6.1.

2.2.2. Lower bounds on the local amplification of the macroscopic field

We display lower bounds on the L^q norm of the gradient fields inside each material that are given in terms of the correctors presented in Theorem 2.6. We begin by presenting a general lower bound that holds for the composition of the sequence $\{\chi_i^\epsilon \nabla u_\epsilon\}_{\epsilon>0}$ with any non-negative Carathéodory function. Recall that $\psi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function if $\psi(x,\cdot)$ is continuous for almost every $x \in \Omega$ and if $\psi(\cdot,\lambda)$ is measurable in x for every $\lambda \in \mathbb{R}^n$. The lower bound on the sequence obtained by the composition of $\psi(x,\cdot)$ with $\chi_i^\epsilon(x)\nabla u_\epsilon(x)$ is given by

Theorem 2.7. *For all Carathéodory functions $\psi \geqslant 0$ and measurable sets $D \subset \Omega$ we have*

$$\int\limits_D \int\limits_Y \psi\big(x, \chi_i(y) p\big(y, \nabla u(x)\big)\big)\,dy\,dx \leqslant \liminf_{\epsilon\to 0} \int\limits_D \psi\big(x, \chi_i^\epsilon(x)\nabla u_\epsilon(x)\big)\,dx.$$

If the sequence $\{\psi(x, \chi_i^\epsilon(x)\nabla u_\epsilon(x))\}_{\epsilon>0}$ is weakly convergent in $L^1(\Omega)$, then the inequality becomes an equality.
In particular, for $\psi(x,\lambda) = |\lambda|^q$ with $q \geqslant 2$, we have

$$\int\limits_D \int\limits_Y \chi_i(y)\big|p\big(y,\nabla u(x)\big)\big|^q\,dy\,dx \leqslant \liminf_{\epsilon\to 0} \int\limits_D \chi_i^\epsilon(x)\big|\nabla u_\epsilon(x)\big|^q\,dx. \tag{2.22}$$

Theorem 2.7 together with (2.22) provide explicit lower bounds on the gradient field inside each material. It relates the local excursions of the gradient inside each phase $\chi_i^\epsilon \nabla u_\epsilon$ to the average gradient ∇u through the multi-scale quantity given by the corrector $p(y,\nabla u(x))$. It is clear from (2.22) that the $L^q(Y\times\Omega;\mathbb{R}^n)$ integrability of $p(y,\nabla u(x))$ provides a lower bound on the $L^q(\Omega;\mathbb{R}^n)$ integrability of ∇u_ϵ.

The proof of Theorem 2.7 is given in Section 6.2.

3. Properties of the homogenized operator *b*

In this section, we prove properties (2.10) and (2.11) of the homogenized operator b. In the rest of the paper, the letter C will represent a generic positive constant independent of ϵ, and it can take different values.

3.1. Proof of (2.11)

Using (2.8) and (2.2), we have

$$\begin{aligned}\big(b(\xi_2)-b(\xi_1),\xi_2-\xi_1\big)&=\int_Y \big(A\big(y,p(y,\xi_2)\big)-A\big(y,p(y,\xi_1)\big),p(y,\xi_2)-p(y,\xi_1)\big)\,dy\\&\geqslant C\bigg(\int_Y \chi_1(y)\big|p(y,\xi_1)-p(y,\xi_2)\big|^{p_1}\,dy+\int_Y \chi_2(y)\big|p(y,\xi_1)-p(y,\xi_2)\big|^{p_2}\,dy\bigg)\\&\geqslant 0.\end{aligned}$$

3.2. Proof of (2.10)

By (2.1), Hölder's inequality, and (2.2) we have

$$\begin{aligned}\big|b(\xi_1)-b(\xi_2)\big|&\leqslant \int_Y \big|A\big(y,p(y,\xi_1)\big)-A\big(y,p(y,\xi_2)\big)\big|\,dy\\&\leqslant C\bigg(\int_Y \chi_1(y)\big|p(y,\xi_1)-p(y,\xi_2)\big|^{p_1}\,dy\bigg)^{\frac{1}{p_1}}\bigg(\int_Y \chi_1(y)\big(1+\big|p(y,\xi_1)\big|+\big|p(y,\xi_2)\big|\big)^{q_2(p_1-2)}\,dy\bigg)^{\frac{1}{q_2}}\\&\quad+C\bigg(\int_Y \chi_2(y)\big|p(y,\xi_1)-p(y,\xi_2)\big|^{p_2}\,dy\bigg)^{\frac{1}{p_2}}\\&\quad\times\bigg(\int_Y \chi_2(y)\big(1+\big|p(y,\xi_1)\big|+\big|p(y,\xi_2)\big|\big)^{q_1(p_2-2)}\,dy\bigg)^{\frac{1}{q_1}}\\&\leqslant C\bigg[\int_Y \big(A\big(y,p(y,\xi_1)\big)-A\big(y,p(y,\xi_2)\big),p(y,\xi_1)-p(y,\xi_2)\big)\,dy\bigg]^{\frac{1}{p_1}}\\&\quad\times\bigg[\int_Y \chi_1(y)\big(1+\big|p(y,\xi_1)\big|+\big|p(y,\xi_2)\big|\big)^{q_2(p_1-2)}\,dy\bigg]^{\frac{1}{q_2}}\\&\quad+C\bigg[\int_Y \big(A\big(y,p(y,\xi_1)\big)-A\big(y,p(y,\xi_2)\big),p(y,\xi_1)-p(y,\xi_2)\big)\,dy\bigg]^{\frac{1}{p_2}}\\&\quad\times\bigg[\int_Y \chi_2(y)\big(1+\big|p(y,\xi_1)\big|+\big|p(y,\xi_2)\big|\big)^{q_1(p_2-2)}\,dy\bigg]^{\frac{1}{q_1}}.\end{aligned}\tag{3.1}$$

Using (3.1), (2.8), (2.6), the Cauchy–Schwarz inequality, Lemma 5.1, and Young's inequality we obtain

$$\begin{aligned}&\leqslant C\bigg[\bigg(\frac{\delta^{p_1}}{p_1}+\frac{\delta^{p_2}}{p_2}\bigg)\big|b(\xi_1)-b(\xi_2)\big|\\&\quad+\frac{\delta^{-q_2}|\xi_1-\xi_2|^{\frac{1}{p_1-1}}(1+|\xi_1|^{p_1}+|\xi_2|^{p_1}+|\xi_1|^{p_2}+|\xi_2|^{p_2})^{\frac{p_1-2}{p_1-1}}}{q_2}\\&\quad+\frac{\delta^{-q_1}|\xi_1-\xi_2|^{\frac{1}{p_2-1}}(1+|\xi_1|^{p_1}+|\xi_2|^{p_1}+|\xi_1|^{p_2}+|\xi_2|^{p_2})^{\frac{p_2-2}{p_2-1}}}{q_1}\bigg].\end{aligned}$$

Rearranging the terms in (3.1), and taking δ small enough we obtain (2.10).

4. Higher order integrability of the homogenized solution

In this section we display higher integrability results for the field gradients inside dispersed micro-structures and layered materials. For dispersions of inclusions, the included material is taken to have a lower power-law exponent than that of the host phase. For both of these cases it is shown that the homogenized solution lies in $W_0^{1,p_2}(\Omega)$. In the following sections we will apply these facts to establish strong approximations for the sequences $\{\chi_i^\epsilon \nabla u_\epsilon\}_{\epsilon>0}$ in $L^{p_2}(\Omega,\mathbb{R}^n)$. The approach taken here is variational and uses the *homogenized Lagrangian* associated with $b(\xi)$ defined in (2.6). The integrability of the homogenized solution u of (2.4) is determined by the growth of the homogenized Lagrangian with respect to its argument.

To proceed we introduce the local Lagrangian associated with power-law composites. The Lagrangian corresponding to the problem studied here is given by

$$\tilde{f}(x,\xi) = q(x)|\xi|^{p(x)}, \quad \text{with } q(x) = \frac{\sigma_1}{p_1}\chi_1(x) + \frac{\sigma_2}{p_2}\chi_2(x), \tag{4.1}$$

where $\xi \in \mathbb{R}^n$ and $x \in \Omega \subset \mathbb{R}^n$. Here $\nabla_\xi \tilde{f}(x,\xi) = A(x,\xi)$, where $A(x,\xi)$ is given by (1.1).

We consider the rescaled Lagrangian

$$\tilde{f}_\epsilon(x,\xi) = \tilde{f}\left(\frac{x}{\epsilon},\xi\right) = \frac{\sigma_1}{p_1}\chi_1^\epsilon(x)|\xi|^{p_1} + \frac{\sigma_2}{p_2}\chi_2^\epsilon(x)|\xi|^{p_2}, \tag{4.2}$$

where $\chi_i^\epsilon(x) = \chi_i(x/\epsilon)$, $i = 1,2$, $\xi \in \mathbb{R}^n$, and $x \in \Omega \subset \mathbb{R}^n$.

The Dirichlet problem given by (2.3) is associated with the variational problem given by

$$E_1^\epsilon(f) = \inf_{u \in W_0^{1,p_1}(\Omega)} \left\{ \int_\Omega \tilde{f}_\epsilon(x,\nabla u)\,dx - \langle f,u\rangle \right\}, \tag{4.3}$$

with $f \in W^{-1,q_2}(\Omega)$. Here (2.3) is the Euler equation for (4.3). However, we also consider

$$E_2^\epsilon(f) = \inf_{u \in W_0^{1,p_2}(\Omega)} \left\{ \int_\Omega \tilde{f}_\epsilon(x,\nabla u)\,dx - \langle f,u\rangle \right\}, \tag{4.4}$$

with $f \in W^{-1,q_2}(\Omega)$ (see [24]). Here $\langle\cdot,\cdot\rangle$ is the duality pairing between $W_0^{1,p_1}(\Omega)$ and $W^{-1,q_2}(\Omega)$.

From [26], we have $\lim_{\epsilon\to 0} E_i^\epsilon = E_i$, for $i = 1,2$, where

$$E_i = \inf_{u \in W_0^{1,p_i}(\Omega)} \left\{ \int_\Omega \hat{\hat{f}}_i\big(\nabla u(x)\big)\,dx - \langle f,u\rangle \right\}. \tag{4.5}$$

In (4.5), $\hat{\hat{f}}_i(\xi)$ is given by

$$\hat{\hat{f}}_i(\xi) = \inf_{v \text{ in } W_{per}^{1,p_i}(Y)} \int_Y \tilde{f}\big(y,\xi + \nabla v(y)\big)\,dy \tag{4.6}$$

and satisfies

$$-c_0 + c_1|\xi|^{p_1} \leqslant \hat{\hat{f}}_i(\xi) \leqslant c_2|\xi|^{p_2} + c_0. \tag{4.7}$$

In general (see [25]), Lavrentiev phenomenon can occur and $E_1 < E_2$. However, for periodic dispersed and layered micro-structures, no Lavrentiev phenomenon occurs and we have the following homogenization theorem.

Theorem 4.1. *For periodic dispersed and layered micro-structures, the homogenized Dirichlet problems satisfy* $E_1 = E_2$, *where* $\hat{\hat{f}} = \hat{\hat{f}}_1 = \hat{\hat{f}}_2$ *and* $c_2 + c_1|\xi|^{p_2} \leqslant \hat{\hat{f}}(\xi)$. *Moreover,* $\nabla_\xi \hat{\hat{f}}(\xi) = b(\xi)$, *where b is the homogenized operator* (2.6).

Proof. Theorem 4.1 has been proved for dispersed periodic media in [26]. We prove Theorem 4.1 for layers following the steps outlined in [26].

We first show that $\hat{\hat{f}} = \hat{\hat{f}}_1 = \hat{\hat{f}}_2$ holds for layered media. Then we show that the homogenized Lagrangian $\hat{\hat{f}}$ satisfies the estimate given by

$$-c_0 + c_1|\xi|^{p_2} \leqslant \hat{\hat{f}}(\xi) \leqslant c_2|\xi|^{p_2} + c_0 \tag{4.8}$$

with $c_0 \geqslant 0$, and $c_1, c_2 > 0$.

We introduce the space of functions $W^{1,p_2}_*(R_2)$ that belong to $W^{1,p_2}(R_2)$ and are periodic on $\partial R_2 \cap \partial Y$.

To prove that $\hat{\hat{f}}_1 = \hat{\hat{f}}_2$, it suffices to show that for every $v \in W^{1,p_1}_{per}(Y)$ satisfying

$$\int_Y \tilde{f}\big(y, \xi + \nabla v(y)\big)\, dy < \infty \tag{4.9}$$

there exists a sequence $v_\epsilon \in W^{1,p_2}_{per}(Y)$ such that

$$\lim_{\epsilon \to 0} \int_Y \tilde{f}\big(y, \xi + \nabla v_\epsilon(y)\big)\, dy = \int_Y \tilde{f}\big(y, \xi + \nabla v(y)\big)\, dy.$$

Let $v \in W^{1,p_1}_{per}(Y)$ that satisfies (4.9). From (4.1) we see that the restriction of v to R_2, denoted by $R(v)$, belongs to $W^{1,p_2}_*(R_2)$. Now we extend $R(v)$ to R_1 so that the extension $\tilde{v}$ belongs to $W^{1,p_2}_{per}(Y)$ and $\tilde{v}(y) = R(v(y)) = v(y)$ on R_2. For future reference we denote the left component of R_2 by $R_{2,L}$ and its boundary with R_1 by Γ_L and similarly denote the right component of R_2 by $R_{2,R}$ with boundary Γ_R. We extend $R(v)$ by reflection across each component of $\Gamma = \Gamma_L \cup \Gamma_R$ into R_1. Denote each of these reflections by v_L and v_R. Next introduce the smooth functions φ_L and φ_R, with $\varphi_L(y) = 1$ for $y \in R_{2,L}$ and compact support in R_1 and $\varphi_R = 1$ for $y \in R_{2,R}$ with compact support in R_1. Here the support sets of φ_L and φ_R do not intersect. The extension is given by

$$\tilde{v}(y) = \begin{cases} \varphi_L(y) v_L(y), & y \text{ in } \operatorname{supp}\{\varphi_L\}, \\ \varphi_R(y) v_R(y), & y \text{ in } \operatorname{supp}\{\varphi_R\}, \\ v(y), & y \text{ in } R_2. \end{cases}$$

Set $z = v - \tilde{v}$. It is clear that $z \in W^{1,p_1}(R_1)$, is periodic on opposite faces of $\partial Y \cap \partial R_1$, zero on Γ and we write

$$\int_Y \tilde{f}\big(y, \xi + \nabla v(y)\big)\, dy = \int_{R_2} f_2\big(\xi + \nabla v(y)\big)\, dy + \int_{R_1} f_1\big(\xi + \nabla \tilde{v}(y) + \nabla z(y)\big)\, dy,$$

where $f_1(\xi) = \frac{\sigma_1}{p_1} |\xi|^{p_1}$ and $f_2(\xi) = \frac{\sigma_2}{p_2} |\xi|^{p_2}$.

We can choose a sequence $\{z_\epsilon\}_{\epsilon > 0} \in C^\infty_0(R_1)$ such that z_ϵ vanishes in R_2 and $z_\epsilon \to z$ in $W^{1,p_1}(R_1)$.

Define $v_\epsilon \in W^{1,p_2}_{per}(Y)$ by

$$v_\epsilon = \begin{cases} v & \text{in } R_2, \\ \tilde{v} + z_\epsilon & \text{in } R_1. \end{cases}$$

Since $v_\epsilon \to v$ in $W^{1,p_1}_{per}(Y)$, we see that

$$\begin{aligned} \lim_{\epsilon \to 0} \int_Y \tilde{f}\big(y, \xi + \nabla v_\epsilon(y)\big)\, dy &= \lim_{\epsilon \to 0} \left(\int_{R_2} f_2\big(\xi + \nabla v(y)\big)\, dy + \int_{R_1} f_1\big(\xi + \nabla \tilde{v}(y) + \nabla z_\epsilon(y)\big)\, dy \right) \\ &= \int_Y \tilde{f}\big(y, \xi + \nabla v(y)\big)\, dy. \end{aligned}$$

Therefore $\hat{\hat{f}} = \hat{\hat{f}}_1 = \hat{\hat{f}}_2$ for layered media.

We establish (4.8) by introducing the convex conjugate of $\hat{\hat{f}}$. We denote the convex dual of $\hat{\hat{f}}_i(\xi)$ by $\hat{\hat{g}}_i(\xi)$; i.e., $\hat{\hat{g}}_i(\xi) = \sup_{\lambda \in \mathbb{R}^n} \{\xi \cdot \lambda - \hat{\hat{f}}_i(\lambda)\}$. It is easily verified (see [24]) that

$$\hat{\hat{g}}_i(\xi) = \inf_{w \text{ in } Sol^{q_i}(Y)} \int_Y \tilde{g}\big(y, \xi + w(y)\big)\, dy \tag{4.10}$$

and

$$-c_0 + c^*_1 |\xi|^{q_1} \leqslant \hat{\hat{g}}_i(\xi) \leqslant c^*_2 |\xi|^{q_2} + c_0. \tag{4.11}$$

Here $Sol^{q_i}(Y)$ are the solenoidal vector fields belonging to $L^{q_i}(Y, \mathbb{R}^n)$ and having mean value zero

$$Sol^{q_i}(Y) = \big\{ w \in L^{q_i}\big(Y; \mathbb{R}^n\big) \colon \operatorname{div} w = 0,\ w \cdot n \text{ anti-periodic} \big\}.$$

We will show that $\hat{\hat{g}} = \hat{\hat{g}}_1 = \hat{\hat{g}}_2$ satisfies $\hat{\hat{g}}(\xi) \leqslant c_2 |\xi|^{q_1} + c_1$, and apply duality to recover $\hat{\hat{f}}(\xi) \geqslant c^*_2 |\xi|^{p_2} + c^*_1$.

To get the upper bound on $\hat{\hat{g}}$ we use the following lemma.

Lemma 4.2. *There exists* τ *with* $\operatorname{div}\tau = 0$ *in* Y*, such that* $\tau \cdot n$ *is anti-periodic on the boundary of* Y*,* $\tau = -\xi$ *in* R_1*, and*

$$\int_Y \left|\tau(y)\right|^{q_1} dy \leqslant C|\xi|^{q_1}.$$

Proof. Let the function $\varphi \in W_*^{1,p_2}(R_2)$ be the solution of

$$\begin{cases} \nabla\varphi|\nabla\varphi|^{p-2} \cdot n \text{ is anti-periodic} & \text{on } \partial R_2 \cap \partial Y, \\ \Delta_{p_2}\varphi = 0 & \text{in } R_2, \\ \left(\nabla\varphi|\nabla\varphi|^{p_2-2} \cdot n\right)\big|_2 = (-\xi \cdot n)|_1 & \text{on } \Gamma, \end{cases}$$

where the subscript 1 indicates the trace on the R_1 side of Γ and 2 indicates the trace on the R_2 side of Γ. The Neumann problem given above is the stationarity condition for the energy $\int_{R_2} |\nabla\phi|^{p_2}\,dx - \int_\Gamma \phi\xi \cdot n\,dS$ when minimized over all $\phi \in W_*^{1,p_2}(R_2)$. The solution of the Neumann problem is unique up to a constant. Here the anti-periodic boundary condition on $\nabla\varphi|\nabla\varphi|^{p-2} \cdot n$ is the natural boundary condition for the problem.

Now we define τ according to

$$\tau = \begin{cases} -\xi & \text{in } R_1, \\ \nabla\varphi|\nabla\varphi|^{p_2-2} & \text{in } R_2 \end{cases}$$

and it follows that

$$|\tau|^{q_1} = \begin{cases} |\xi|^{q_1} & \text{in } R_1, \\ [(\nabla\varphi|\nabla\varphi|^{p_2-2})^2]^{\frac{q_1}{2}} = (|\nabla\varphi|^{p_2-1})^{q_1} = |\nabla\varphi|^{p_2} & \text{in } R_2. \end{cases} \tag{4.12}$$

Then, for $\psi \in W_*^{1,p_2}(R_2)$ we have

$$\begin{aligned} \int_{R_2} |\nabla\varphi|^{p_2-2}\nabla\varphi \cdot \nabla\psi\,dy &= \int_\Gamma \psi|\nabla\varphi|^{p_2-2}\nabla\varphi \cdot n\,dS + \int_{\partial R_2 \cap \partial Y} \psi|\nabla\varphi|^{p_2-2}\nabla\varphi \cdot n\,dS \\ &= -\int_\Gamma \psi\xi \cdot n\,dS = -\int_{R_2} \nabla\psi \cdot \xi\,dy. \end{aligned} \tag{4.13}$$

Set $\psi = \varphi$ in (4.13) and an application of Hölder's inequality gives

$$\int_{R_2} \left|\nabla\varphi(y)\right|^{p_2} dy \leqslant \int_{R_2} |\xi|^{q_1}\,dy. \tag{4.14}$$

Therefore, using (4.12) and (4.14), we have

$$\begin{aligned} \int_Y \left|\tau(y)\right|^{q_1} dy &= \int_{R_1} \left|\tau(y)\right|^{q_1} dy + \int_{R_2} \left|\tau(y)\right|^{q_1} dy \\ &= \int_{R_1} |\xi|^{q_1}\,dy + \int_{R_2} \left|\nabla\varphi(y)\right|^{p_2} dy \leqslant C|\xi|^{q_1}. \qquad \square \end{aligned}$$

Taking $\hat{\hat{g}}$ to be the conjugate of $\hat{\hat{f}}$, and choosing τ in $Sol^{q_1}(Y)$ as in Lemma 4.2, we obtain

$$\begin{aligned} \hat{\hat{g}}(\xi) &= \inf_{\tau \text{ in } Sol^{q_1}(Y)} \int_Y \tilde{g}(y, \xi + \tau)\,dy \leqslant \int_Y \tilde{g}(y, \xi + \tau)\,dy \\ &\leqslant \int_{R_1} \tilde{g}(y, 0)\,dy + \int_{R_2} \tilde{g}(y, \xi + \tau)\,dy \leqslant c_1 + c_2 \int_{R_2} |\xi + \tau|^{q_1}\,dy \leqslant c_1 + c_2|\xi|^{q_1}, \end{aligned}$$

and the left-hand inequality in (4.8) follows from duality.

This concludes the proof of Theorem 4.1. □

Collecting results we now prove Theorem 2.5. Indeed the minimizer of E_1 is precisely the solution u of (2.4) and (2.5). Theorem 4.1 establishes the coercivity of E_1 over $W_0^{1,p_2}(\Omega)$, thus the solution u lies in $W_0^{1,p_2}(\Omega)$.

5. Some useful lemmas and estimates

In this section we state and prove a priori bounds and convergence properties for the sequences p_ϵ defined in (2.14), ∇u_ϵ, and $A_\epsilon(x, p_\epsilon(x, \nabla u_\epsilon))$ that are used in the proof of the main results of this paper.

Lemma 5.1. *For every $\xi \in \mathbb{R}^n$ we have*

$$\int_Y \chi_1(y)\big|p(y,\xi)\big|^{p_1}\,dy + \int_Y \chi_2(y)\big|p(y,\xi)\big|^{p_2}\,dy \leqslant C\big(1 + |\xi|^{p_1}\theta_1 + |\xi|^{p_2}\theta_2\big), \tag{5.1}$$

and by a change of variables, we obtain

$$\int_{Y_\epsilon} \chi_1^\epsilon(x)\big|p_\epsilon(x,\xi)\big|^{p_1}\,dx + \int_{Y_\epsilon} \chi_2^\epsilon(x)\big|p_\epsilon(x,\xi)\big|^{p_2}\,dx \leqslant C\big(1 + |\xi|^{p_1}\theta_1 + |\xi|^{p_2}\theta_2\big)|Y_\epsilon|. \tag{5.2}$$

Proof. Let $\xi \in \mathbb{R}^n$. By (2.2) we have that

$$\big(A\big(y, p(y,\xi)\big), p(y,\xi)\big) \geqslant C\big(\chi_1(y)\big|p(y,\xi)\big|^{p_1} + \chi_2(y)\big|p(y,\xi)\big|^{p_2}\big).$$

Integrating both sides over Y, using (2.1), and Young's inequality, we get

$$\begin{aligned}&\int_Y \chi_1(y)\big|p(y,\xi)\big|^{p_1}\,dy + \int_Y \chi_2(y)\big|p(y,\xi)\big|^{p_2}\,dy\\ &\quad\leqslant C\left[\big(\delta^{q_2}\theta_1 + \delta^{q_1}\theta_2\big) + \left(\frac{|\xi|^{p_1}\theta_1}{\delta^{p_1}} + \frac{|\xi|^{p_2}\theta_2}{\delta^{p_2}}\right) + \big(\delta^{q_2} + \delta^{q_1}\big)\left(\int_Y \chi_1(y)\big|p(y,\xi)\big|^{p_1}\,dy + \int_Y \chi_2(y)\big|p(y,\xi)\big|^{p_2}\,dy\right)\right].\end{aligned}$$

Doing some algebraic manipulations, we obtain

$$\begin{aligned}&\big(1 - C\big(\delta^{q_2} + \delta^{q_1}\big)\big)\left(\int_Y \chi_1(y)\big|p(y,\xi)\big|^{p_1}\,dy + \int_Y \chi_2(y)\big|p(y,\xi)\big|^{p_2}\,dy\right)\\ &\quad\leqslant C\big[\big(\delta^{q_2}\theta_1 + \delta^{q_1}\theta_2\big) + \big(\delta^{-p_1}|\xi|^{p_1}\theta_1 + \delta^{-p_2}|\xi|^{p_2}\theta_2\big)\big].\end{aligned}$$

On choosing an appropriate δ, we finally obtain (5.1). □

Lemma 5.2. *For every $\xi_1, \xi_2 \in \mathbb{R}^n$ we have*

$$\begin{aligned}&\int_Y \chi_1(y)\big|p(y,\xi_1) - p(y,\xi_2)\big|^{p_1}\,dy + \int_Y \chi_2(y)\big|p(y,\xi_1) - p(y,\xi_2)\big|^{p_2}\,dy\\ &\quad\leqslant C\Big[\big(1 + |\xi_1|^{p_1}\theta_1 + |\xi_1|^{p_2}\theta_2 + |\xi_2|^{p_1}\theta_1 + |\xi_2|^{p_2}\theta_2\big)^{\frac{p_1-2}{p_1-1}}|\xi_1 - \xi_2|^{\frac{p_1}{p_1-1}}\theta_1^{\frac{1}{p_1-1}}\\ &\qquad + \big(1 + |\xi_1|^{p_1}\theta_1 + |\xi_1|^{p_2}\theta_2 + |\xi_2|^{p_1}\theta_1 + |\xi_2|^{p_2}\theta_2\big)^{\frac{p_2-2}{p_2-1}}|\xi_1 - \xi_2|^{\frac{p_2}{p_2-1}}\theta_2^{\frac{1}{p_2-1}}\Big]\end{aligned} \tag{5.3}$$

and by doing a change of variables, we obtain

$$\begin{aligned}&\int_{Y_\epsilon} \chi_1^\epsilon(x)\big|p_\epsilon(x,\xi_1) - p_\epsilon(x,\xi_2)\big|^{p_1}\,dx + \int_{Y_\epsilon} \chi_2^\epsilon(x)\big|p_\epsilon(x,\xi_1) - p_\epsilon(x,\xi_2)\big|^{p_2}\,dx\\ &\quad\leqslant C\Big[\big(1 + |\xi_1|^{p_1}\theta_1 + |\xi_1|^{p_2}\theta_2 + |\xi_2|^{p_1}\theta_1 + |\xi_2|^{p_2}\theta_2\big)^{\frac{p_1-2}{p_1-1}}|\xi_1 - \xi_2|^{\frac{p_1}{p_1-1}}\theta_1^{\frac{1}{p_1-1}}\\ &\qquad + \big(1 + |\xi_1|^{p_1}\theta_1 + |\xi_1|^{p_2}\theta_2 + |\xi_2|^{p_1}\theta_1 + |\xi_2|^{p_2}\theta_2\big)^{\frac{p_2-2}{p_2-1}}|\xi_1 - \xi_2|^{\frac{p_2}{p_2-1}}\theta_2^{\frac{1}{p_2-1}}\Big]|Y_\epsilon|.\end{aligned} \tag{5.4}$$

Proof. By (2.2), (2.8), and (2.1) we have that

$$\begin{aligned}&\int_Y \chi_1(y)\big|p(y,\xi_1) - p(y,\xi_2)\big|^{p_1}\,dy + \int_Y \chi_2(y)\big|p(y,\xi_1) - p(y,\xi_2)\big|^{p_2}\,dy\\ &\quad\leqslant C\int_Y \big|A\big(y, p(y,\xi_1)\big) - A\big(y, p(y,\xi_2)\big)\big||\xi_1 - \xi_2|\,dy\end{aligned}$$

$$\leqslant C\biggl[\int\limits_Y \chi_1(y)\bigl|p(y,\xi_1)-p(y,\xi_2)\bigr|\bigl(1+\bigl|p(y,\xi_1)\bigr|+\bigl|p(y,\xi_2)\bigr|\bigr)^{p_1-2}|\xi_1-\xi_2|\,dy$$

$$+\int\limits_Y \chi_2(y)\bigl|p(y,\xi_1)-p(y,\xi_2)\bigr|\bigl(1+\bigl|p(y,\xi_1)\bigr|+\bigl|p(y,\xi_2)\bigr|\bigr)^{p_2-2}|\xi_1-\xi_2|\,dy\biggr].$$

Using Hölder's inequality in the first term with $r_1=p_1/(p_1-2)$, $r_2=p_1$, $r_3=p_1$, and in the second term with $s_1=p_2/(p_2-2)$, $s_2=p_2$, $s_3=p_2$, and using Lemma 5.1, we obtain

$$\leqslant C\biggl[\bigl(1+|\xi_1|^{p_1}\theta_1+|\xi_1|^{p_2}\theta_2+|\xi_2|^{p_1}\theta_1+|\xi_2|^{p_2}\theta_2\bigr)^{\frac{p_1-2}{p_1}}$$

$$\times|\xi_1-\xi_2|\theta_1^{\frac{1}{p_1}}\biggl(\int\limits_Y \chi_1(y)\bigl|p(y,\xi_1)-p(y,\xi_2)\bigr|^{p_1}\,dy\biggr)^{\frac{1}{p_1}}$$

$$+\bigl(1+|\xi_1|^{p_1}\theta_1+|\xi_1|^{p_2}\theta_2+|\xi_2|^{p_1}\theta_1+|\xi_2|^{p_2}\theta_2\bigr)^{\frac{p_2-2}{p_2}}$$

$$\times|\xi_1-\xi_2|\theta_2^{\frac{1}{p_2}}\biggl(\int\limits_Y \chi_2(y)\bigl|p(y,\xi_1)-p(y,\xi_2)\bigr|^{p_2}\,dy\biggr)^{\frac{1}{p_2}}\biggr].$$

By Young's inequality, we get

$$\leqslant C\biggl[\frac{\delta^{-q_2}(1+|\xi_1|^{p_1}\theta_1+|\xi_1|^{p_2}\theta_2+|\xi_2|^{p_1}\theta_1+|\xi_2|^{p_2}\theta_2)^{\frac{(p_1-2)q_2}{p_1}}|\xi_1-\xi_2|^{q_2}\theta_1^{\frac{q_2}{p_1}}}{q_2}$$

$$+\frac{\delta^{p_1}\int_Y\chi_1(y)|p(y,\xi_1)-p(y,\xi_2)|^{p_1}\,dy}{p_1}+\frac{\delta^{p_2}\int_Y\chi_2(y)|p(y,\xi_1)-p(y,\xi_2)|^{p_2}\,dy}{p_2}$$

$$+\frac{\delta^{-q_1}(1+|\xi_1|^{p_1}\theta_1+|\xi_1|^{p_2}\theta_2+|\xi_2|^{p_1}\theta_1+|\xi_2|^{p_2}\theta_2)^{\frac{(p_2-2)q_1}{p_2}}|\xi_1-\xi_2|^{q_1}\theta_2^{\frac{q_1}{p_2}}}{q_1}\biggr].$$

Straightforward algebraic manipulation delivers

$$k_\delta\biggl(\int\limits_Y \chi_1(y)\bigl|p(y,\xi_1)-p(y,\xi_2)\bigr|^{p_1}\,dy+\int\limits_Y \chi_2(y)\bigl|p(y,\xi_1)-p(y,\xi_2)\bigr|^{p_2}\,dy\biggr)$$

$$\leqslant C\biggl[\frac{\delta^{-q_2}(1+|\xi_1|^{p_1}\theta_1+|\xi_1|^{p_2}\theta_2+|\xi_2|^{p_1}\theta_1+|\xi_2|^{p_2}\theta_2)^{\frac{p_1-2}{p_1-1}}|\xi_1-\xi_2|^{\frac{p_1}{p_1-1}}\theta_1^{\frac{1}{p_1-1}}}{q_2}$$

$$+\frac{\delta^{-q_1}(1+|\xi_1|^{p_1}\theta_1+|\xi_1|^{p_2}\theta_2+|\xi_2|^{p_1}\theta_1+|\xi_2|^{p_2}\theta_2)^{\frac{p_2-2}{p_2-1}}|\xi_1-\xi_2|^{\frac{p_2}{p_2-1}}\theta_2^{\frac{1}{p_2-1}}}{q_1}\biggr],$$

where $k_\delta=\min\{(1-\frac{C\delta^{p_1}}{p_1}),(1-\frac{C\delta^{p_2}}{p_2})\}$.

The result follows on choosing δ small enough so that k_δ is positive. □

Lemma 5.3. *Let φ be such that*

$$\sup_{\epsilon>0}\biggl\{\int\limits_\Omega \chi_1^\epsilon(x)\bigl|\varphi(x)\bigr|^{p_1}\,dx+\int\limits_\Omega \chi_2^\epsilon(x)\bigl|\varphi(x)\bigr|^{p_2}\,dx\biggr\}<\infty,$$

and let Ψ be a simple function of the form

$$\Psi(x)=\sum_{j=0}^{m}\eta_j\chi_{\Omega_j}(x), \tag{5.5}$$

with $\eta_j\in\mathbb{R}^n\setminus\{0\}$, $\Omega_j\subset\subset\Omega$, $|\partial\Omega_j|=0$, $\Omega_j\cap\Omega_k=\emptyset$ for $j\neq k$ and $j,k=1,\dots,m$; and set $\eta_0=0$ and $\Omega_0=\Omega\setminus\bigcup_{j=1}^m\Omega_j$. Then

$$
\begin{aligned}
&\limsup_{\epsilon\to 0}\Big(\int_\Omega \chi_1^\epsilon(x)\big|p_\epsilon\big(x, M_\epsilon\varphi(x)\big)-p_\epsilon\big(x,\Psi(x)\big)\big|^{p_1}\,dx+\int_\Omega \chi_2^\epsilon(x)\big|p_\epsilon\big(x, M_\epsilon\varphi(x)\big)-p_\epsilon\big(x,\Psi(x)\big)\big|^{p_2}\,dx\Big)\\
&\quad\leqslant \limsup_{\epsilon\to 0} C\sum_{i=1}^{2}\Bigg[\Big(|\Omega|+\int_\Omega \chi_1^\epsilon(x)\big|\varphi(x)\big|^{p_1}\,dx+\int_\Omega \chi_2^\epsilon(x)\big|\varphi(x)\big|^{p_2}\,dx\\
&\qquad+\int_\Omega \chi_1^\epsilon(x)\big|\Psi(x)\big|^{p_1}\,dx+\int_\Omega \chi_2^\epsilon(x)\big|\Psi(x)\big|^{p_2}\,dx\Big)^{\frac{p_i-2}{p_i-1}}\Big(\int_\Omega \chi_i^\epsilon(x)\big|\varphi(x)-\Psi(x)\big|^{p_i}\,dx\Big)^{\frac{1}{p_i-1}}\Bigg].
\end{aligned}\tag{5.6}
$$

Proof. Let Ψ be of the form (5.5). For every $\epsilon>0$, let us denote $\Omega_\epsilon=\bigcup_{i\in I_\epsilon}\overline{Y_\epsilon^i}$; and for $j=0,1,2,\dots,m$, we set

$$
I_\epsilon^j=\big\{i\in I_\epsilon\colon\ Y_\epsilon^i\subseteq\Omega_j\big\}\quad\text{and}\quad J_\epsilon^j=\big\{i\in I_\epsilon\colon\ Y_\epsilon^i\cap\Omega_j\neq\emptyset,\ Y_\epsilon^i\setminus\Omega_j\neq\emptyset\big\}.
$$

Furthermore, $E_\epsilon^j=\bigcup_{i\in I_\epsilon^j}\overline{Y_\epsilon^i}$, $F_\epsilon^j=\bigcup_{i\in J_\epsilon^j}\overline{Y_\epsilon^i}$, and as $\epsilon\to 0$, we have $|F_\epsilon^j|\to 0$.

Set

$$
\xi_\epsilon^i=\frac{1}{|Y_\epsilon^i|}\int_{Y_\epsilon^i}\varphi(y)\,dy.
$$

For ϵ sufficiently small Ω_j ($j\neq 0$) is contained in Ω_ϵ.

From (5.5), (2.20), using the fact that $\Omega_j\subset E_\epsilon^j\cup F_\epsilon^j$, Lemma 5.2, and Hölder's inequality it follows that

$$
\begin{aligned}
&\int_\Omega \chi_1^\epsilon(x)\big|p_\epsilon(x,M_\epsilon\varphi)-p_\epsilon(x,\Psi)\big|^{p_1}\,dx+\int_\Omega \chi_2^\epsilon(x)\big|p_\epsilon(x,M_\epsilon\varphi)-p_\epsilon(x,\Psi)\big|^{p_2}\,dx\\
&\quad\leqslant C\Bigg[\Big(|\Omega|+\int_\Omega \chi_1^\epsilon(x)|M_\epsilon\varphi-\varphi|^{p_1}\,dx+\int_\Omega \chi_1^\epsilon(x)|\varphi|^{p_1}\,dx+\int_\Omega \chi_2^\epsilon(x)|M_\epsilon\varphi-\varphi|^{p_2}\,dx\\
&\qquad+\int_\Omega \chi_2^\epsilon(x)\big|\varphi(x)\big|^{p_2}\,dx+\int_\Omega \chi_1^\epsilon(x)\big|\Psi(x)\big|^{p_1}\,dx+\int_\Omega \chi_2^\epsilon(x)\big|\Psi(x)\big|^{p_2}\,dx\Big)^{\frac{p_1-2}{p_1-1}}\\
&\qquad\times\Big(\int_\Omega \chi_1^\epsilon(x)|M_\epsilon\varphi-\varphi|^{p_1}\,dx+\int_\Omega \chi_1^\epsilon(x)|\varphi-\Psi|^{p_1}\,dx\Big)^{\frac{1}{p_1-1}}\\
&\qquad+\Big(|\Omega|+\int_\Omega \chi_1^\epsilon(x)|M_\epsilon\varphi-\varphi|^{p_1}\,dx+\int_\Omega \chi_1^\epsilon(x)|\varphi|^{p_1}\,dx+\int_\Omega \chi_2^\epsilon(x)|M_\epsilon\varphi-\varphi|^{p_2}\,dx\\
&\qquad+\int_\Omega \chi_2^\epsilon(x)\big|\varphi(x)\big|^{p_2}\,dx+\int_\Omega \chi_1^\epsilon(x)\big|\Psi(x)\big|^{p_1}\,dx+\int_\Omega \chi_2^\epsilon(x)\big|\Psi(x)\big|^{p_2}\,dx\Big)^{\frac{p_2-2}{p_2-1}}\\
&\qquad\times\Big(\int_\Omega \chi_2^\epsilon(x)|M_\epsilon\varphi-\varphi|^{p_2}\,dx+\int_\Omega \chi_2^\epsilon(x)|\varphi-\Psi|^{p_2}\,dx\Big)^{\frac{1}{p_2-1}}\Bigg]+C\sum_{j=0}^{m}\Bigg[\Big(\int_{F_\epsilon^j}\theta_1\Big|\sum_{i\in J_\epsilon^j}\chi_{Y_\epsilon^i}(x)\xi_\epsilon^i-\eta_j\Big|^{p_1}\,dx\Big)^{\frac{1}{p_1-1}}\\
&\qquad\times\Big(\int_{F_\epsilon^j}\big|M_\epsilon\varphi(x)\big|^{p_1}\theta_1\,dx+\big|F_\epsilon^j\big|+|\eta_j|^{p_1}\theta_1\big|F_\epsilon^j\big|+|\eta_j|^{p_2}\theta_2\big|F_\epsilon^j\big|+\int_{F_\epsilon^j}\big|M_\epsilon\varphi(x)\big|^{p_2}\theta_2\,dx\Big)^{\frac{p_1-2}{p_1-1}}\\
&\qquad+\Big(\big|F_\epsilon^j\big|+\int_{F_\epsilon^j}\big|M_\epsilon\varphi(x)\big|^{p_1}\theta_1\,dx+\int_{F_\epsilon^j}\big|M_\epsilon\varphi(x)\big|^{p_2}\theta_2\,dx+|\eta_j|^{p_1}\theta_1\big|F_\epsilon^j\big|+|\eta_j|^{p_2}\theta_2\big|F_\epsilon^j\big|\Big)^{\frac{p_2-2}{p_2-1}}\\
&\qquad\times\Big(\int_{F_\epsilon^j}\theta_2\Big|\sum_{i\in J_\epsilon^j}\chi_{Y_\epsilon^i}(x)\xi_\epsilon^i-\eta_j\Big|^{p_2}\,dx\Big)^{\frac{1}{p_2-1}}\Bigg].
\end{aligned}\tag{5.7}
$$

Since $|\partial\Omega_j| = 0$ for $j \neq 0$, we have that $|F_\epsilon^j| \to 0$ as $\epsilon \to 0$, for every $j = 0, 1, 2, \dots, m$. By property (1) of M_ϵ mentioned in Section 2.2.1, we have

$$\int_\Omega \chi_i^\epsilon(x)\left|M_\epsilon\varphi(x) - \varphi(x)\right|^{p_i} dx \to 0, \quad \text{as } \epsilon \to 0, \text{ for } i = 1, 2.$$

Therefore, taking $\limsup$ as $\epsilon \to 0$ in (5.7), we obtain (5.6). □

Lemma 5.4. *If the micro-structure is dispersed or layered, we have that*

$$\sup_{\epsilon>0}\left\{\int_\Omega \chi_i^\epsilon(x)\left|p_\epsilon\big(x, M_\epsilon\nabla u(x)\big)\right|^{p_i} dx\right\} \leqslant C < \infty, \quad \textit{for } i = 1, 2.$$

Proof. Using (2.20), we have

$$\begin{aligned}
&\int_\Omega \chi_1^\epsilon(x)\left|p_\epsilon\big(x, M_\epsilon\nabla u(x)\big)\right|^{p_1} dx + \int_\Omega \chi_2^\epsilon(x)\left|p_\epsilon\big(x, M_\epsilon\nabla u(x)\big)\right|^{p_2} dx \\
&\quad = \sum_{i\in\mathbf{I}_\epsilon}\left[\int_{Y_\epsilon^i} \chi_1^\epsilon(x)\left|p_\epsilon\big(x, \xi_\epsilon^i\big)\right|^{p_1} dx + \int_{Y_\epsilon^i} \chi_2^\epsilon(x)\left|p_\epsilon\big(x, \xi_\epsilon^i\big)\right|^{p_2} dx\right] \\
&\quad \leqslant C\sum_{i\in\mathbf{I}_\epsilon}\big(1 + \left|\xi_\epsilon^i\right|^{p_1}\theta_1 + \left|\xi_\epsilon^i\right|^{p_2}\theta_2\big)\left|Y_\epsilon^i\right| \\
&\quad = C\sum_{i\in\mathbf{I}_\epsilon}\big(\left|Y_\epsilon^i\right| + \left|\xi_\epsilon^i\right|^{p_1}\theta_1\left|Y_\epsilon^i\right| + \left|\xi_\epsilon^i\right|^{p_2}\theta_2\left|Y_\epsilon^i\right|\big) \\
&\quad \leqslant C\big(|\Omega| + \|\nabla u\|_{\mathbf{L}^{p_1}(\Omega)}^{p_1} + \|\nabla u\|_{\mathbf{L}^{p_2}(\Omega)}^{p_2}\big) < \infty,
\end{aligned}$$

where the last three inequalities follow from Lemma 5.1, Jensen's inequality, and Theorem 2.5. □

Lemma 5.5. *Let u_ϵ be the solution to* (2.3). *Then* (2.9) *holds.*

Proof. Evaluating u_ϵ in the weak formulation for (2.3), applying Hölder's inequality, and since $f \in W^{-1,q_2}(\Omega)$, we obtain

$$\begin{aligned}
\int_\Omega \big(A_\epsilon(x, \nabla u_\epsilon), \nabla u_\epsilon\big)\, dx &= \sigma_1\int_\Omega \chi_1^\epsilon(x)|\nabla u_\epsilon|^{p_1} dx + \sigma_2\int_\Omega \chi_2^\epsilon(x)|\nabla u_\epsilon|^{p_2} dx \\
&= \langle f, u_\epsilon\rangle \leqslant C\left[\left(\int_\Omega \chi_1^\epsilon(x)|\nabla u_\epsilon|^{p_1} dx\right)^{\frac{1}{p_1}} + \left(\int_\Omega \chi_2^\epsilon(x)|\nabla u_\epsilon|^{p_2} dx\right)^{\frac{1}{p_2}}\right].
\end{aligned} \tag{5.8}$$

Applying Young's inequality to the last term in (5.8), we obtain

$$\begin{aligned}
&\sigma_1\int_\Omega \chi_1^\epsilon(x)|\nabla u_\epsilon|^{p_1} dx + \sigma_2\int_\Omega \chi_2^\epsilon(x)|\nabla u_\epsilon|^{p_2} dx \\
&\quad \leqslant C\left[\frac{\delta^{p_1}}{p_1}\int_\Omega \chi_1^\epsilon(x)|\nabla u_\epsilon|^{p_1} dx + \frac{\delta^{-q_2}}{q_2} + \frac{\delta^{p_2}}{p_2}\int_\Omega \chi_2^\epsilon(x)|\nabla u_\epsilon|^{p_2} dx + \frac{\delta^{-q_1}}{q_1}\right].
\end{aligned} \tag{5.9}$$

By rearranging the terms in (5.9), one gets

$$\left(\sigma_1 - C\frac{\delta^{p_1}}{p_1}\right)\int_\Omega \chi_1^\epsilon(x)|\nabla u_\epsilon|^{p_1} dx + \left(\sigma_2 - C\frac{\delta^{p_2}}{p_2}\right)\int_\Omega \chi_2^\epsilon(x)|\nabla u_\epsilon|^{p_2} dx \leqslant \frac{\delta^{-q_2}}{q_2} + \frac{\delta^{-q_1}}{q_1}.$$

Therefore, by choosing δ small enough so that $\min\{\sigma_1 - C\frac{\delta^{p_1}}{p_1}, \sigma_2 - C\frac{\delta^{p_2}}{p_2}\}$ is positive, one obtains

$$\int_\Omega \chi_1^\epsilon(x)\left|\nabla u_\epsilon(x)\right|^{p_1} dx + \int_\Omega \chi_2^\epsilon(x)\left|\nabla u_\epsilon(x)\right|^{p_2} dx \leqslant C. \qquad \square$$

Lemma 5.6. *For all $j = 0, \dots, m$, we have that $\int_{\Omega_j} |(A_\epsilon(x, p_\epsilon(x, \eta_j)), \nabla u_\epsilon(x))|\,dx$ and $\int_{\Omega_j} |(A_\epsilon(x, \nabla u_\epsilon(x)), p_\epsilon(x, \eta_j))|\,dx$ are uniformly bounded with respect to ϵ.*

Proof. Using Hölder's inequality, (2.1), and (2.9), we obtain

$$\begin{aligned}
&\int_{\Omega_j} \big|\big(A_\epsilon\big(x, p_\epsilon(x, \eta_j)\big), \nabla u_\epsilon(x)\big)\big|\,dx \\
&\quad\leqslant \int_{\Omega_j} \big|A_\epsilon\big(x, p_\epsilon(x, \eta_j)\big)\big|\big|\nabla u_\epsilon(x)\big|\,dx \\
&\quad\leqslant C\bigg[\bigg(\int_{\Omega_j} \chi_1^\epsilon(x)\big(1 + \big|p_\epsilon(x, \eta_j)\big|\big)^{p_1}\,dx\bigg)^{\frac{1}{q_2}} + \bigg(\int_{\Omega_j} \chi_2^\epsilon(x)\big(1 + \big|p_\epsilon(x, \eta_j)\big|\big)^{p_2}\,dx\bigg)^{\frac{1}{q_1}}\bigg] \\
&\quad\leqslant C, \quad \text{where } C \text{ does not depend on } \epsilon.
\end{aligned}$$

The proof of the uniform boundedness of $\int_{\Omega_j} |(A_\epsilon(x, \nabla u_\epsilon(x)), p_\epsilon(x, \eta_j))|\,dx$ follows in the same manner. □

Lemma 5.7. *As $\epsilon \to 0$, up to a subsequence, $(A_\epsilon(\cdot, p_\epsilon(\cdot, \eta_j)), \nabla u_\epsilon(\cdot))$ converges weakly to a function $g_j \in L^1(\Omega_j; \mathbb{R})$, for all $j = 0, \dots, m$. In a similar way, up to a subsequence, $(A_\epsilon(\cdot, \nabla u_\epsilon(\cdot)), p_\epsilon(\cdot, \eta_j))$ converges weakly to a function $h_j \in L^1(\Omega_j; \mathbb{R})$, for all $j = 0, \dots, m$.*

Proof. We prove the first statement of the lemma, the second statement follows in a similar way. The lemma follows from the Dunford–Pettis theorem (see [4]). To apply this theorem we establish the following conditions:

(1) $\int_{\Omega_j} |(A_\epsilon(x, p_\epsilon(x, \eta_j)), \nabla u_\epsilon(x))|\,dx$ is uniformly bounded with respect to ϵ.
(2) For all $j = 0, \dots, m$, $(A_\epsilon(\cdot, p_\epsilon(\cdot, \eta_j)), \nabla u_\epsilon(\cdot))$ is equiintegrable.

The first condition is proved in Lemma 5.6. For the second condition, we have that $\chi_1^\epsilon(\cdot)|A_\epsilon(\cdot, p_\epsilon(\cdot, \eta_j))|^{q_2}$ and $\chi_2^\epsilon(\cdot)|A_\epsilon(\cdot, p_\epsilon(\cdot, \eta_j))|^{q_1}$ are equiintegrable (see for example Theorem 1.5 of [4]).

By (2.9), for any $E \subset \Omega$, we have

$$\max_{i=1,2}\bigg\{\sup_{\epsilon>0}\bigg\{\bigg(\int_E \chi_i^\epsilon(x)\big|\nabla u_\epsilon(x)\big|^{p_i}\,dx\bigg)^{\frac{1}{p_i}}\bigg\}\bigg\} \leqslant C.$$

Let $\alpha > 0$ arbitrary and choose $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $\alpha_1^{1/q_2} + \alpha_2^{1/q_1} < \alpha/C$.

For α_1 and α_2, there exist $\lambda(\alpha_1) > 0$ and $\lambda(\alpha_2) > 0$ such that for every $E \subset \Omega$ with $|E| < \min\{\lambda(\alpha_1), \lambda(\alpha_2)\}$,

$$\int_E \chi_1^\epsilon(x)\big|A_\epsilon\big(x, p_\epsilon(x, \eta_j)\big)\big|^{q_2}\,dx < \alpha_1 \quad \text{and} \quad \int_E \chi_2^\epsilon(x)\big|A_\epsilon\big(x, p_\epsilon(x, \eta_j)\big)\big|^{q_1}\,dx < \alpha_2.$$

Take $\lambda = \lambda(\alpha) = \min\{\lambda(\alpha_1), \lambda(\alpha_2)\}$. Then, for all $E \subset \Omega$ with $|E| < \lambda(\alpha)$, we have

$$\begin{aligned}
\int_E \big|\big(A_\epsilon\big(x, p_\epsilon(x, \eta_j)\big), \nabla u_\epsilon(x)\big)\big|\,dx &\leqslant \int_E \big|A_\epsilon\big(x, p_\epsilon(x, \eta_j)\big)\big|\big|\nabla u_\epsilon(x)\big|\,dx \\
&\leqslant \bigg(\int_E \chi_1^\epsilon(x)\big|A_\epsilon\big(x, p_\epsilon(x, \eta_j)\big)\big|^{q_2}\,dx\bigg)^{\frac{1}{q_2}}\bigg(\int_E \chi_1^\epsilon(x)\big|\nabla u_\epsilon(x)\big|^{p_1}\,dx\bigg)^{\frac{1}{p_1}} \\
&\quad + \bigg(\int_E \chi_2^\epsilon(x)\big|A_\epsilon\big(x, p_\epsilon(x, \eta_j)\big)\big|^{q_1}\,dx\bigg)^{\frac{1}{q_1}}\bigg(\int_E \chi_2^\epsilon(x)\big|\nabla u_\epsilon(x)\big|^{p_2}\,dx\bigg)^{\frac{1}{p_2}} \\
&\leqslant C\big(\alpha_1^{1/q_2} + \alpha_2^{1/q_1}\big) < \alpha,
\end{aligned}$$

for every $\alpha > 0$, and so $(A_\epsilon(\cdot, p_\epsilon(\cdot, \eta_j)), \nabla u_\epsilon(\cdot))$ is equiintegrable. □

6. Proof of the main results

6.1. Proof of the corrector theorem

We are now in the position to give the proof of Theorem 2.6.

Proof of Theorem 2.6. Let $u_\epsilon \in W_0^{1,p_1}(\Omega)$ the solutions of (2.3). By (2.2), we have that

$$\int_\Omega \big[\chi_1^\epsilon(x)\big|p_\epsilon\big(x, M_\epsilon\nabla u(x)\big) - \nabla u_\epsilon(x)\big|^{p_1} + \chi_2^\epsilon(x)\big|p_\epsilon\big(x, M_\epsilon\nabla u(x)\big) - \nabla u_\epsilon(x)\big|^{p_2}\big]\,dx$$
$$\leqslant C\int_\Omega \big(A_\epsilon\big(x, p_\epsilon\big(x, M_\epsilon\nabla u(x)\big)\big) - A_\epsilon\big(x, \nabla u_\epsilon(x)\big), p_\epsilon\big(x, M_\epsilon\nabla u(x)\big) - \nabla u_\epsilon(x)\big)\,dx.$$

To prove Theorem 2.6, we show that

$$\int_\Omega \big(A_\epsilon\big(x, p_\epsilon\big(x, M_\epsilon\nabla u(x)\big)\big) - A_\epsilon\big(x, \nabla u_\epsilon(x)\big), p_\epsilon\big(x, M_\epsilon\nabla u(x)\big) - \nabla u_\epsilon(x)\big)\,dx$$
$$= \int_\Omega \big(A_\epsilon\big(x, p_\epsilon(x, M_\epsilon\nabla u)\big), p_\epsilon(x, M_\epsilon\nabla u)\big)\,dx - \int_\Omega \big(A_\epsilon\big(x, p_\epsilon(x, M_\epsilon\nabla u)\big), \nabla u_\epsilon\big)\,dx$$
$$- \int_\Omega \big(A_\epsilon(x, \nabla u_\epsilon), p_\epsilon(x, M_\epsilon\nabla u)\big)\,dx + \int_\Omega \big(A_\epsilon(x, \nabla u_\epsilon), \nabla u_\epsilon\big)\,dx$$

goes to 0, as $\epsilon \to 0$. This is done in four steps.

In what follows, we use the following notation

$$\xi_\epsilon^i = \frac{1}{|Y_\epsilon^i|}\int_{Y_\epsilon^i} \nabla u\,dx.$$

Step 1. Let us prove that

$$\int_\Omega \big(A_\epsilon\big(x, p_\epsilon(x, M_\epsilon\nabla u)\big), p_\epsilon(x, M_\epsilon\nabla u)\big)\,dx \to \int_\Omega \big(b(\nabla u), \nabla u\big)\,dx \tag{6.1}$$

as $\epsilon \to 0$.

Proof. From (2.13) and (2.20), we obtain

$$\int_\Omega \big(A_\epsilon\big(x, p_\epsilon\big(x, M_\epsilon\nabla u(x)\big)\big), p_\epsilon\big(x, M_\epsilon\nabla u(x)\big)\big)\,dx$$
$$= \int_{\Omega_\epsilon} \big(A_\epsilon\big(x, p_\epsilon\big(x, M_\epsilon\nabla u(x)\big)\big), p_\epsilon\big(x, M_\epsilon\nabla u(x)\big)\big)\,dx$$
$$= \sum_{i\in I_\epsilon}\int_{Y_\epsilon^i} \left(A\left(\frac{x}{\epsilon}, p\left(\frac{x}{\epsilon}, \xi_\epsilon^i\right)\right), p\left(\frac{x}{\epsilon}, \xi_\epsilon^i\right)\right)dx$$
$$= \epsilon^n\sum_{i\in I_\epsilon}\int_Y \big(A\big(y, p\big(y, \xi_\epsilon^i\big)\big), p\big(y, \xi_\epsilon^i\big)\big)\,dy$$
$$= \sum_{i\in I_\epsilon}\int_\Omega \chi_{Y_\epsilon^i}(x)\big(b\big(\xi_\epsilon^i\big), \xi_\epsilon^i\big)\,dx$$
$$= \int_\Omega \big(b\big(M_\epsilon\nabla u(x)\big), M_\epsilon\nabla u(x)\big)\,dx.$$

By (2.10), the definition of q_1, and Hölder's inequality we have

$$\int_\Omega \bigl|b\bigl(M_\epsilon \nabla u(x)\bigr) - b\bigl(\nabla u(x)\bigr)\bigr|^{q_1}\,dx$$

$$\leqslant C\biggl[\biggl(\int_\Omega \bigl|M_\epsilon \nabla u(s) - \nabla u(s)\bigr|^{p_2}\,dx\biggr)^{\frac{1}{(p_2-1)^2}} + \biggl(\int_\Omega \bigl|M_\epsilon \nabla u(x) - \nabla u(x)\bigr|^{p_2}\,dx\biggr)^{\frac{1}{(p_2-1)(p_1-1)}}\biggr].$$

From property (1) of M_ϵ, we obtain that

$$b(M_\epsilon \nabla u) \to b(\nabla u) \quad \text{in } L^{q_1}\bigl(\Omega;\mathbb{R}^n\bigr),\ \text{as } \epsilon \to 0. \tag{6.2}$$

Now, (6.1) follows from (6.2) since $M_\epsilon \nabla u \to \nabla u$ in $L^{p_2}(\Omega;\mathbb{R}^n)$, so

$$\int_\Omega \bigl(A_\epsilon\bigl(x, p_\epsilon\bigl(x, M_\epsilon \nabla u(x)\bigr)\bigr), p_\epsilon\bigl(x, M_\epsilon \nabla u(x)\bigr)\bigr)\,dx$$

$$= \int_\Omega \bigl(b\bigl(M_\epsilon \nabla u(x)\bigr), M_\epsilon \nabla u(x)\bigr)\,dx \to \int_\Omega \bigl(b\bigl(\nabla u(x)\bigr), \nabla u(x)\bigr)\,dx,$$

as $\epsilon \to 0$. □

Step 2. We now show that

$$\int_\Omega \bigl(A_\epsilon\bigl(x, p_\epsilon\bigl(x, M_\epsilon \nabla u(x)\bigr)\bigr), \nabla u_\epsilon(x)\bigr)\,dx \to \int_\Omega \bigl(b\bigl(\nabla u(x)\bigr), \nabla u(x)\bigr)\,dx \tag{6.3}$$

as $\epsilon \to 0$.

Proof. Let $\delta > 0$. From Theorem 2.5 we have $\nabla u \in L^{p_2}(\Omega;\mathbb{R}^n)$ and there exists a simple function Ψ satisfying the assumptions of Lemma 5.3 such that

$$\|\nabla u - \Psi\|_{L^{p_2}(\Omega;\mathbb{R}^n)} \leqslant \delta. \tag{6.4}$$

Let us write

$$\int_\Omega \bigl(A_\epsilon\bigl(x, p_\epsilon\bigl(x, M_\epsilon \nabla u(x)\bigr)\bigr), \nabla u_\epsilon(x)\bigr)\,dx$$

$$= \int_\Omega \bigl(A_\epsilon\bigl(x, p_\epsilon(x, \Psi)\bigr), \nabla u_\epsilon\bigr)\,dx + \int_\Omega \bigl(A_\epsilon\bigl(x, p_\epsilon(x, M_\epsilon \nabla u)\bigr) - A_\epsilon\bigl(x, p_\epsilon(x, \Psi)\bigr), \nabla u_\epsilon\bigr)\,dx.$$

We first show that

$$\int_\Omega \bigl(A_\epsilon\bigl(x, p_\epsilon\bigl(x, \Psi(x)\bigr)\bigr), \nabla u_\epsilon(x)\bigr)\,dx \to \int_\Omega \bigl(b\bigl(\Psi(x)\bigr), \nabla u(x)\bigr)\,dx, \quad \text{as } \epsilon \to 0.$$

We have

$$\int_\Omega \bigl(A_\epsilon\bigl(x, p_\epsilon\bigl(x, \Psi(x)\bigr)\bigr), \nabla u_\epsilon(x)\bigr)\,dx = \sum_{j=0}^{m} \int_{\Omega_j} \bigl(A_\epsilon\bigl(x, p_\epsilon(x, \eta_j)\bigr), \nabla u_\epsilon(x)\bigr)\,dx.$$

Now from (2.19), we have that $A_\epsilon(\cdot, p_\epsilon(\cdot, \eta_j)) \rightharpoonup b(\eta_j) \in L^{q_2}(\Omega_j;\mathbb{R}^n)$, and by (2.12), $\int_{\Omega_j} (A_\epsilon(x, p_\epsilon(x, \eta_j)), \nabla\varphi(x))\,dx = 0$, for $\varphi \in W_0^{1,p_1}(\Omega_j)$.

Take $\varphi = \delta u_\epsilon$, with $\delta \in C_0^\infty(\Omega_j)$ to get

$$0 = \int_{\Omega_j} \bigl(A_\epsilon\bigl(x, p_\epsilon(x, \eta_j)\bigr), (\nabla\delta) u_\epsilon\bigr)\,dx + \int_{\Omega_j} \bigl(A_\epsilon\bigl(x, p_\epsilon(x, \eta_j)\bigr), (\nabla u_\epsilon)\delta\bigr)\,dx.$$

Taking the limit as $\epsilon \to 0$, and using the fact that $u^\epsilon \rightharpoonup u$ in $W_0^{1,p_1}(\Omega)$ and (2.19), we have by Lemma 5.7 that

$$\int_{\Omega_j} g_j(x)\delta(x)\,dx = \lim_{\epsilon\to 0}\int_{\Omega_j}\bigl(A_\epsilon\bigl(x, p_\epsilon(x,\eta_j)\bigr), (\nabla u_\epsilon)\delta\bigr)\,dx = \int_{\Omega_j}\bigl(b(\eta_j), (\nabla u)\delta\bigr)\,dx.$$

Therefore, we may conclude that $g_j = (b(\eta_j), \nabla u)$, so

$$\sum_{j=0}^{n}\int_{\Omega_j}\bigl(A_\epsilon\bigl(x, p_\epsilon(x,\eta_j)\bigr), \nabla u_\epsilon(x)\bigr)\,dx \to \sum_{j=0}^{n}\int_{\Omega_j}\bigl(b(\eta_j), \nabla u(x)\bigr)\,dx, \quad \text{as } \epsilon \to 0.$$

Thus, we get

$$\int_{\Omega}\bigl(A_\epsilon\bigl(x, p_\epsilon\bigl(x, \Psi(x)\bigr)\bigr), \nabla u_\epsilon(x)\bigr)\,dx \to \int_{\Omega}\bigl(b\bigl(\Psi(x)\bigr), \nabla u(x)\bigr)\,dx, \quad \text{as } \epsilon \to 0.$$

On the other hand, let us estimate

$$\int_{\Omega}\bigl(A_\epsilon\bigl(x, p_\epsilon\bigl(x, M_\epsilon\nabla u(x)\bigr)\bigr) - A_\epsilon\bigl(x, p_\epsilon\bigl(x, \Psi(x)\bigr)\bigr), \nabla u_\epsilon(x)\bigr)\,dx.$$

By (2.1) and Hölder's inequality we obtain

$$\begin{aligned}
&\left|\int_{\Omega}\bigl(A_\epsilon\bigl(x, p_\epsilon\bigl(x, M_\epsilon\nabla u(x)\bigr)\bigr) - A_\epsilon\bigl(x, p_\epsilon\bigl(x, \Psi(x)\bigr)\bigr), \nabla u_\epsilon(x)\bigr)\,dx\right| \\
&\quad\leqslant C\biggl(\int_{\Omega}\chi_1^\epsilon(x)\bigl|p_\epsilon(x, M_\epsilon\nabla u) - p_\epsilon(x,\Psi)\bigr|^{p_1}\,dx\biggr)^{\frac{1}{p_1}}\biggl(\int_{\Omega}\chi_1^\epsilon(x)|\nabla u_\epsilon|^{p_1}\,dx\biggr)^{\frac{1}{p_1}} \\
&\qquad\times\biggl(\int_{\Omega}\chi_1^\epsilon(x)\bigl(1 + \bigl|p_\epsilon(x, M_\epsilon\nabla u)\bigr|^{p_1} + \bigl|p_\epsilon(x,\Psi)\bigr|^{p_1}\bigr)\,dx\biggr)^{\frac{p_1-2}{p_1}} \\
&\qquad+ C\biggl(\int_{\Omega}\chi_2^\epsilon(x)\bigl|p_\epsilon(x, M_\epsilon\nabla u) - p_\epsilon(x,\Psi)\bigr|^{p_2}\,dx\biggr)^{\frac{1}{p_2}}\biggl(\int_{\Omega}\chi_2^\epsilon(x)|\nabla u_\epsilon|^{p_2}\,dx\biggr)^{\frac{1}{p_2}} \\
&\qquad\times\biggl(\int_{\Omega}\chi_2^\epsilon(x)\bigl(1 + \bigl|p_\epsilon(x, M_\epsilon\nabla u)\bigr|^{p_2} + \bigl|p_\epsilon(x,\Psi)\bigr|^{p_2}\bigr)\,dx\biggr)^{\frac{p_2-2}{p_2}}.
\end{aligned} \tag{6.5}$$

Applying (2.9), (5.4), and Lemma 5.1 to the right-hand side of (6.5), we obtain

$$\begin{aligned}
&\left|\int_{\Omega}\bigl(A_\epsilon\bigl(x, p_\epsilon\bigl(x, M_\epsilon\nabla u(x)\bigr)\bigr) - A_\epsilon\bigl(x, p_\epsilon\bigl(x, \Psi(x)\bigr)\bigr), \nabla u_\epsilon(x)\bigr)\,dx\right| \\
&\quad\leqslant C\biggl[\biggl(\int_{\Omega}\chi_1^\epsilon(x)\bigl|p_\epsilon\bigl(x, M_\epsilon\nabla u(x)\bigr) - p_\epsilon\bigl(x, \Psi(x)\bigr)\bigr|^{p_1}\,dx\biggr)^{\frac{1}{p_1}} \\
&\qquad+\biggl(\int_{\Omega}\chi_2^\epsilon(x)\bigl|p_\epsilon\bigl(x, M_\epsilon\nabla u(x)\bigr) - p_\epsilon\bigl(x, \Psi(x)\bigr)\bigr|^{p_2}\,dx\biggr)^{\frac{1}{p_2}}\biggr].
\end{aligned} \tag{6.6}$$

Applying Lemma 5.3 and (6.4) to (6.6), we discover that

$$\begin{aligned}
&\limsup_{\epsilon\to 0}\left|\int_{\Omega}\bigl(A_\epsilon\bigl(x, p_\epsilon\bigl(x, M_\epsilon\nabla u(x)\bigr)\bigr) - A_\epsilon\bigl(x, p_\epsilon\bigl(x, \Psi(x)\bigr)\bigr), \nabla u_\epsilon(x)\bigr)\,dx\right| \\
&\quad\leqslant C\bigl[\bigl(\delta^{q_1} + \delta^{q_2}\bigr)^{\frac{1}{p_1}} + \bigl(\delta^{q_1} + \delta^{q_2}\bigr)^{\frac{1}{p_2}}\bigr],
\end{aligned} \tag{6.7}$$

where C is independent of δ. Since δ is arbitrary we conclude that the limit on the left-hand side of (6.7) is equal to 0.

Finally, using the continuity of b and Hölder's inequality we obtain

$$\left|\int_\Omega \big(b\big(\nabla u(x)\big) - b\big(\Psi(x)\big), \nabla u(x)\big)\,dx\right| \leqslant C\big[\delta^{\frac{q_1}{p_1-1}} + \delta^{\frac{q_1}{p_2-1}}\big]^{\frac{1}{q_1}},$$

where C does not depend on δ.

Step 2 is proved noticing that δ can be taken arbitrarily small. □

Step 3. We will show that

$$\int_\Omega \big(A_\epsilon\big(x, \nabla u_\epsilon(x)\big), p_\epsilon\big(x, M_\epsilon \nabla u(x)\big)\big)\,dx \to \int_\Omega \big(b\big(\nabla u(x)\big), \nabla u(x)\big)\,dx \tag{6.8}$$

as $\epsilon \to 0$.

Proof. Let $\delta > 0$. As in the proof of Step 2, assume Ψ is a simple function satisfying assumptions of Lemma 5.3 and such that $\|\nabla u - \Psi\|_{L^{p_2}(\Omega;\mathbb{R}^n)} < \delta$.

Let us write

$$\begin{aligned}&\int_\Omega \big(A_\epsilon\big(x, \nabla u_\epsilon(x)\big), p_\epsilon\big(x, M_\epsilon \nabla u(x)\big)\big)\,dx \\ &\quad = \int_\Omega \big(A_\epsilon\big(x, \nabla u_\epsilon(x)\big), p_\epsilon\big(x, \Psi(x)\big)\big)\,dx + \int_\Omega \big(A_\epsilon\big(x, \nabla u_\epsilon(x)\big), p_\epsilon\big(x, M_\epsilon \nabla u(x)\big) - p_\epsilon\big(x, \Psi(x)\big)\big)\,dx.\end{aligned}$$

We first show that

$$\int_\Omega \big(A_\epsilon\big(x, \nabla u_\epsilon(x)\big), p_\epsilon\big(x, \Psi(x)\big)\big)\,dx \to \int_\Omega \big(b\big(\nabla u(x)\big), \Psi(x)\big)\,dx.$$

We start by writing

$$\int_\Omega \big(A_\epsilon\big(x, \nabla u_\epsilon(x)\big), p_\epsilon\big(x, \Psi(x)\big)\big)\,dx = \sum_{j=0}^{m} \int_{\Omega_j} \big(A_\epsilon\big(x, \nabla u_\epsilon(x)\big), p_\epsilon(x, \eta_j)\big)\,dx.$$

From Lemma 5.7, up to a subsequence, $(A_\epsilon(\cdot, \nabla u_\epsilon), p_\epsilon(\cdot, \eta_j))$ converges weakly to a function $h_j \in L^1(\Omega_j; \mathbb{R})$, as $\epsilon \to 0$. By Theorem 2.1, we have $A_\epsilon(\cdot, \nabla u_\epsilon) \rightharpoonup b(\nabla u) \in L^{q_2}(\Omega; \mathbb{R}^n)$ and

$$-\operatorname{div}\big(A_\epsilon(x, \nabla u_\epsilon)\big) = f = -\operatorname{div}\big(b(\nabla u)\big).$$

From (2.17), p_ϵ satisfies $p_\epsilon(\cdot, \eta_j) \rightharpoonup \eta_j$ in $L^{p_1}(\Omega_j, \mathbb{R}^n)$.

Arguing as in Step 2, we find that $(A_\epsilon(x, \nabla u_\epsilon(x)), p_\epsilon(x, \eta_j)) \rightharpoonup (b(\nabla u(x)), \eta_j)$ in $D'(\Omega_j)$, as $\epsilon \to 0$.

Therefore, we may conclude that $h_j = (b(\nabla u), \eta_j)$, and hence,

$$\sum_{j=0}^{n} \int_{\Omega_j} \big(A_\epsilon\big(x, \nabla u_\epsilon(x)\big), p_\epsilon(x, \eta_j)\big)\,dx \to \sum_{j=0}^{n} \int_{\Omega_j} \big(b\big(\nabla u(x)\big), \eta_j\big)\,dx, \quad \text{as } \epsilon \to 0.$$

Thus, we get

$$\int_\Omega \big(A_\epsilon\big(x, \nabla u_\epsilon(x)\big), p_\epsilon\big(x, \Psi(x)\big)\big)\,dx \to \int_\Omega \big(b\big(\nabla u(x)\big), \Psi(x)\big)\,dx, \quad \text{as } \epsilon \to 0.$$

Moreover, applying Hölder's inequality and (2.1) we have

$$\begin{aligned}&\left|\int_\Omega \big(A_\epsilon\big(x, \nabla u_\epsilon(x)\big), p_\epsilon\big(x, M_\epsilon \nabla u(x)\big) - p_\epsilon\big(x, \Psi(x)\big)\big)\,dx\right| \\ &\quad \leqslant C\bigg[\bigg(\int_\Omega \chi_1^\epsilon\big(1 + |\nabla u_\epsilon|\big)^{p_1}\bigg)^{\frac{1}{q_2}} \bigg(\int_\Omega \chi_1^\epsilon \big|p_\epsilon(x, M_\epsilon \nabla u) - p_\epsilon(x, \Psi)\big|^{p_1}\,dx\bigg)^{\frac{1}{p_1}} \\ &\qquad + \bigg(\int_\Omega \chi_2^\epsilon\big(1 + |\nabla u_\epsilon|\big)^{p_2}\bigg)^{\frac{1}{q_1}} \bigg(\int_\Omega \chi_2^\epsilon \big|p_\epsilon(x, M_\epsilon \nabla u) - p_\epsilon(x, \Psi)\big|^{p_2}\,dx\bigg)^{\frac{1}{p_2}}\bigg].\end{aligned}$$

As in the proof of Step 2 we see that

$$\limsup_{\epsilon\to 0}\left|\int_\Omega \big(A_\epsilon(x,\nabla u_\epsilon),\, p_\epsilon(x,M_\epsilon\nabla u)-p_\epsilon(x,\Psi)\big)\,dx\right| \leqslant C\big[\big(\delta^{q_2}+\delta^{q_1}\big)^{\frac{1}{p_1}}+\big(\delta^{q_1}+\delta^{q_2}\big)^{\frac{1}{p_2}}\big],$$

where C does not depend on δ.

Hence, proceeding as in Step 2, we find that

$$\limsup_{\epsilon\to 0}\left|\int_\Omega \big(A_\epsilon(x,\nabla u_\epsilon),\, p_\epsilon(x,M_\epsilon\nabla u)\big)\,dx-\int_\Omega \big(b(\nabla u),\nabla u\big)\,dx\right|$$
$$\leqslant C\Big(\big(\delta^{q_2}+\delta^{q_1}\big)^{\frac{1}{p_1}}+\big(\delta^{q_2}+\delta^{q_1}\big)^{\frac{1}{p_2}}+0+\delta\big\|b(\nabla u)\big\|_{L^{q_2}(\Omega,\mathbb{R}^n)}\Big),$$

where C is independent of δ. Now since δ is arbitrarily small, the proof of Step 3 is complete. □

Step 4. Finally, let us prove that

$$\int_\Omega \big(A_\epsilon\big(x,\nabla u_\epsilon(x)\big),\nabla u_\epsilon(x)\big)\,dx \to \int_\Omega \big(b\big(\nabla u(x)\big),\nabla u(x)\big)\,dx, \quad \text{as } \epsilon\to 0. \tag{6.9}$$

Proof. Since

$$\int_\Omega \big(A_\epsilon(x,\nabla u_\epsilon),\nabla u_\epsilon\big)\,dx=\big\langle -\operatorname{div}\big(A_\epsilon(x,\nabla u_\epsilon)\big),u_\epsilon\big\rangle=\langle f,u_\epsilon\rangle, \tag{6.10}$$

$$\int_\Omega \big(b(\nabla u),\nabla u\big)\,dx=\big\langle -\operatorname{div}\big(b(\nabla u)\big),u\big\rangle=\langle f,u\rangle, \tag{6.11}$$

and $u_\epsilon \rightharpoonup u$ in $W^{1,p_1}(\Omega)$, the result follows immediately. □

Finally, Theorem 2.6 follows from (6.1), (6.3), (6.8) and (6.9). □

6.2. Proof of the lower bound on the amplification of the macroscopic field by the micro-structure

The sequence $\{\chi_i^\epsilon(x)\nabla u_\epsilon(x)\}_{\epsilon>0}$ has a Young measure $\nu^i=\{\nu_x^i\}_{x\in\Omega}$ associated to it (see Theorem 6.2 and the discussion following in [14]), for $i=1,2$.

As a consequence of Theorem 2.6 proved in the previous section, we have that

$$\left\|\chi_i^\epsilon(x)p\left(\frac{x}{\epsilon},M_\epsilon(\nabla u)(x)\right)-\chi_i^\epsilon(x)\nabla u_\epsilon(x)\right\|_{\mathbf{L}^{p_i}(\Omega;\mathbb{R}^n)}\to 0,$$

as $\epsilon\to 0$, which implies that the sequences

$$\left\{\chi_i^\epsilon(x)p\left(\frac{x}{\epsilon},M_\epsilon(\nabla u)(x)\right)\right\}_{\epsilon>0} \quad\text{and}\quad \big\{\chi_i^\epsilon(x)\nabla u_\epsilon(x)\big\}_{\epsilon>0}$$

share the same Young measure (see Lemma 6.3 of [14]), for $i=1,2$.

The next lemma identifies the Young measure ν^i.

Lemma 6.1. *For all $\phi\in C_0(\mathbb{R}^n)$ and for all $\zeta\in C_0^\infty(\mathbb{R}^n)$, we have*

$$\int_\Omega \zeta(x)\int_{\mathbf{R}^n}\phi(\lambda)\,d\nu_x^i(\lambda)\,dx=\int_\Omega \zeta(x)\int_Y \phi\big(\chi_i(y)p\big(y,\nabla u(x)\big)\big)\,dy\,dx. \tag{6.12}$$

Proof. To prove (6.12), we will show that given $\phi\in C_0(\mathbb{R}^n)$ and $\zeta\in C_0^\infty(\mathbb{R}^n)$,

$$\lim_{\epsilon\to 0}\int_\Omega \zeta(x)\phi\left(\chi_i^\epsilon(x)p\left(\frac{x}{\epsilon},M_\epsilon(\nabla u)(x)\right)\right)dx=\int_\Omega \zeta(x)\int_Y \phi\big(\chi_i(y)p\big(y,\nabla u(x)\big)\big)\,dy\,dx. \tag{6.13}$$

We consider the difference

$$\begin{aligned}
&\left|\int\limits_{\Omega} \zeta(x)\phi\left(\chi_i\left(\frac{x}{\epsilon}\right)p\left(\frac{x}{\epsilon}, M_\epsilon(\nabla u)(x)\right)\right)dx - \int\limits_{\Omega}\zeta(x)\int\limits_{Y}\phi\big(\chi_i(y)p\big(y,\nabla u(x)\big)\big)\,dy\,dx\right| \\
&\quad\leqslant \left|\sum_{i\in I_\epsilon}\int\limits_{Y_\epsilon^i}\zeta(x)\phi\left(\chi_i\left(\frac{x}{\epsilon}\right)p\left(\frac{x}{\epsilon},\xi_\epsilon^i\right)\right)dx - \int\limits_{\Omega_\epsilon}\zeta(x)\int\limits_{Y}\phi\big(\chi_i(y)p\big(y,\nabla u(x)\big)\big)\,dy\,dx\right| \\
&\qquad + C|\Omega\setminus\Omega_\epsilon|.
\end{aligned}\tag{6.14}$$

Note that the term $C|\Omega\setminus\Omega_\epsilon|$ goes to 0, as $\epsilon\to 0$. Now set x_ϵ^i to be the center of Y_ϵ^i. On the first integral use the change of variables $x = x_\epsilon^i + \epsilon y$, where y belongs to Y, and since $dx = \epsilon^n\,dy$, we get

$$\begin{aligned}
&\left|\sum_{i\in I_\epsilon}\int\limits_{Y_\epsilon^i}\zeta(x)\phi\left(\chi_i\left(\frac{x}{\epsilon}\right)p\left(\frac{x}{\epsilon},\xi_\epsilon^i\right)\right)dx - \sum_{i\in I_\epsilon}\int\limits_{Y_\epsilon^i}\zeta(x)\int\limits_{Y}\phi\big(\chi_i(y)p\big(y,\nabla u(x)\big)\big)\,dy\,dx\right| \\
&\quad= \left|\sum_{i\in I_\epsilon}\epsilon^n\int\limits_{Y}\zeta\big(x_\epsilon^i+\epsilon y\big)\phi\big(\chi_i(y)p\big(y,\xi_\epsilon^i\big)\big)\,dy - \sum_{i\in I_\epsilon}\int\limits_{Y_\epsilon^i}\zeta(x)\int\limits_{Y}\phi\big(\chi_i(y)p\big(y,\nabla u(x)\big)\big)\,dy\,dx\right|.
\end{aligned}$$

Applying Taylor's expansion for ζ, we have

$$\begin{aligned}
&\leqslant\left|\sum_{i\in I_\epsilon}\int\limits_{Y_\epsilon^i}\int\limits_{Y}\big(\zeta(x)+CO(\epsilon)\big)\big[\phi\big(\chi_i(y)p\big(y,\xi_\epsilon^i\big)\big)-\phi\big(\chi_i(y)p\big(y,\nabla u(x)\big)\big)\big]dy\,dx\right| \\
&\quad + CO(\epsilon) \\
&\leqslant\left|\int\limits_{\Omega_\epsilon}\big|\zeta(x)\big|\int\limits_{Y}\big|\phi\big(\chi_i(y)p\big(y,M_\epsilon\nabla u(x)\big)\big)-\phi\big(\chi_i(y)p\big(y,\nabla u(x)\big)\big)\big|\,dy\,dx\right| \\
&\quad + CO(\epsilon).
\end{aligned}$$

Because of the uniform Lipschitz continuity of ϕ, we get

$$\leqslant C\left|\int\limits_{\Omega_\epsilon}\big|\zeta(x)\big|\int\limits_{Y}\big|p\big(y,M_\epsilon\nabla u(x)\big)-p\big(y,\nabla u(x)\big)\big|\,dy\,dx\right| + CO(\epsilon).$$

By Hölder's inequality twice and Lemma 5.2, we have

$$\begin{aligned}
&\leqslant C\Bigg\{\bigg(\int\limits_{\Omega_\epsilon}\big|\zeta(x)\big|^{q_2}\,dx\bigg)^{1/q_2}\Bigg[\int\limits_{\Omega_\epsilon}\Big(\big|M_\epsilon\nabla u(x)-\nabla u(x)\big|^{\frac{p_1}{p_1-1}}\theta_1^{\frac{1}{p_1-1}} \\
&\quad\times\big(1+\big|M_\epsilon\nabla u(x)\big|^{p_1}\theta_1+\big|M_\epsilon\nabla u(x)\big|^{p_2}\theta_2+\big|\nabla u(x)\big|^{p_1}\theta_1+\big|\nabla u(x)\big|^{p_2}\theta_2\big)^{\frac{p_1-2}{p_1-1}} \\
&\quad+\big|M_\epsilon\nabla u(x)-\nabla u(x)\big|^{\frac{p_2}{p_2-1}}\theta_2^{\frac{1}{p_2-1}} \\
&\quad\times\big(1+\big|M_\epsilon\nabla u(x)\big|^{p_1}\theta_1+\big|M_\epsilon\nabla u(x)\big|^{p_2}\theta_2+\big|\nabla u(x)\big|^{p_1}\theta_1+\big|\nabla u(x)\big|^{p_2}\theta_2\big)^{\frac{p_2-2}{p_2-1}}\Big)\,dx\Bigg]^{1/p_1} \\
&\quad+\bigg(\int\limits_{\Omega_\epsilon}\big|\zeta(x)\big|^{q_1}\,dx\bigg)^{1/q_1}\Bigg[\int\limits_{\Omega_\epsilon}\Big(\big|M_\epsilon\nabla u(x)-\nabla u(x)\big|^{\frac{p_1}{p_1-1}}\theta_1^{\frac{1}{p_1-1}} \\
&\quad\times\big(1+\big|M_\epsilon\nabla u(x)\big|^{p_1}\theta_1+\big|M_\epsilon\nabla u(x)\big|^{p_2}\theta_2+\big|\nabla u(x)\big|^{p_1}\theta_1+\big|\nabla u(x)\big|^{p_2}\theta_2\big)^{\frac{p_1-2}{p_1-1}} \\
&\quad+\big|M_\epsilon\nabla u(x)-\nabla u(x)\big|^{\frac{p_2}{p_2-1}}\theta_2^{\frac{1}{p_2-1}} \\
&\quad\times\big(1+\big|M_\epsilon\nabla u(x)\big|^{p_1}\theta_1+\big|M_\epsilon\nabla u(x)\big|^{p_2}\theta_2+\big|\nabla u(x)\big|^{p_1}\theta_1+\big|\nabla u(x)\big|^{p_2}\theta_2\big)^{\frac{p_2-2}{p_2-1}}\Big)\,dx\Bigg]^{1/p_2}\Bigg\} \\
&\quad+CO(\epsilon).
\end{aligned}$$

Applying Hölder's inequality again, we get

$$\begin{aligned}
&\leqslant C\Biggl[\biggl(\int\limits_{\Omega_\epsilon}\bigl|M_\epsilon\nabla u(x)-\nabla u(x)\bigr|^{p_1}\,dx\biggr)^{\frac{1}{p_1-1}}\\
&\quad+\biggl(\int\limits_{\Omega_\epsilon}\bigl|M_\epsilon\nabla u(x)-\nabla u(x)\bigr|^{p_2}\,dx\biggr)^{\frac{1}{p_2-1}}\Biggr]^{1/p_1}\\
&\quad+C\Biggl[\biggl(\int\limits_{\Omega_\epsilon}\bigl|M_\epsilon\nabla u(x)-\nabla u(x)\bigr|^{p_1}\,dx\biggr)^{\frac{1}{p_1-1}}\\
&\quad+\biggl(\int\limits_{\Omega_\epsilon}\bigl|M_\epsilon\nabla u(x)-\nabla u(x)\bigr|^{p_2}\,dx\biggr)^{\frac{1}{p_2-1}}\Biggr]^{1/p_2}+CO(\epsilon).
\end{aligned}$$

Finally, from the approximation property of M_ϵ in Section 2.2.1, as $\epsilon \to 0$, we obtain (6.13).

Therefore, from Proposition 4.4 of [15] and (6.13) we have

$$\begin{aligned}
\int\limits_{\Omega}\zeta(x)\int\limits_{\mathbf{R}^n}\phi(\lambda)\,d\nu_x^i(\lambda)\,dx &= \int\limits_{\Omega}\zeta(x)\int\limits_{Y}\phi\bigl(\chi_i(y)p\bigl(y,\nabla u(x)\bigr)\bigr)\,dy\,dx\\
&=\lim_{\epsilon\to 0}\int\limits_{\Omega}\zeta(x)\phi\biggl(\chi_i^\epsilon(x)p\biggl(\frac{x}{\epsilon},M_\epsilon(\nabla u)(x)\biggr)\biggr)\,dx\\
&\leqslant\lim_{\epsilon\to 0}\int\limits_{\Omega}\zeta(x)\phi\bigl(\chi_i^\epsilon(x)\nabla u_\epsilon(x)\bigr)\,dx,
\end{aligned}$$

for all $\phi \in C_0(\mathbb{R}^n)$ and for all $\zeta \in C_0^\infty(\mathbb{R}^n)$. $\square$

The proof of Theorem 2.7 follows from Lemma 6.1 and Theorem 6.11 in [14].

7. Summary

In this paper we consider a composite material made from two materials with different power-law behavior. The exponent of the power law is different for each material and taken to be p_1 in material one and p_2 in material two with $2 \leqslant p_1 < p_2 < \infty$. For this case we have introduced a corrector theory for the strong approximation of fields inside these composites, see Theorem 2.6. The correctors are then used to provide lower bounds on the local singularity strength inside micro-structured media. The bounds are multi-scale in nature and quantify the amplification of applied macroscopic fields by the micro-structure, see Theorem 2.7. These results are shown to hold for finely mixed periodic dispersions of inclusions and for layers. Future work seeks to extend the analysis to multi-phase power-law materials and for different regimes of exponents p_1 and p_2.

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